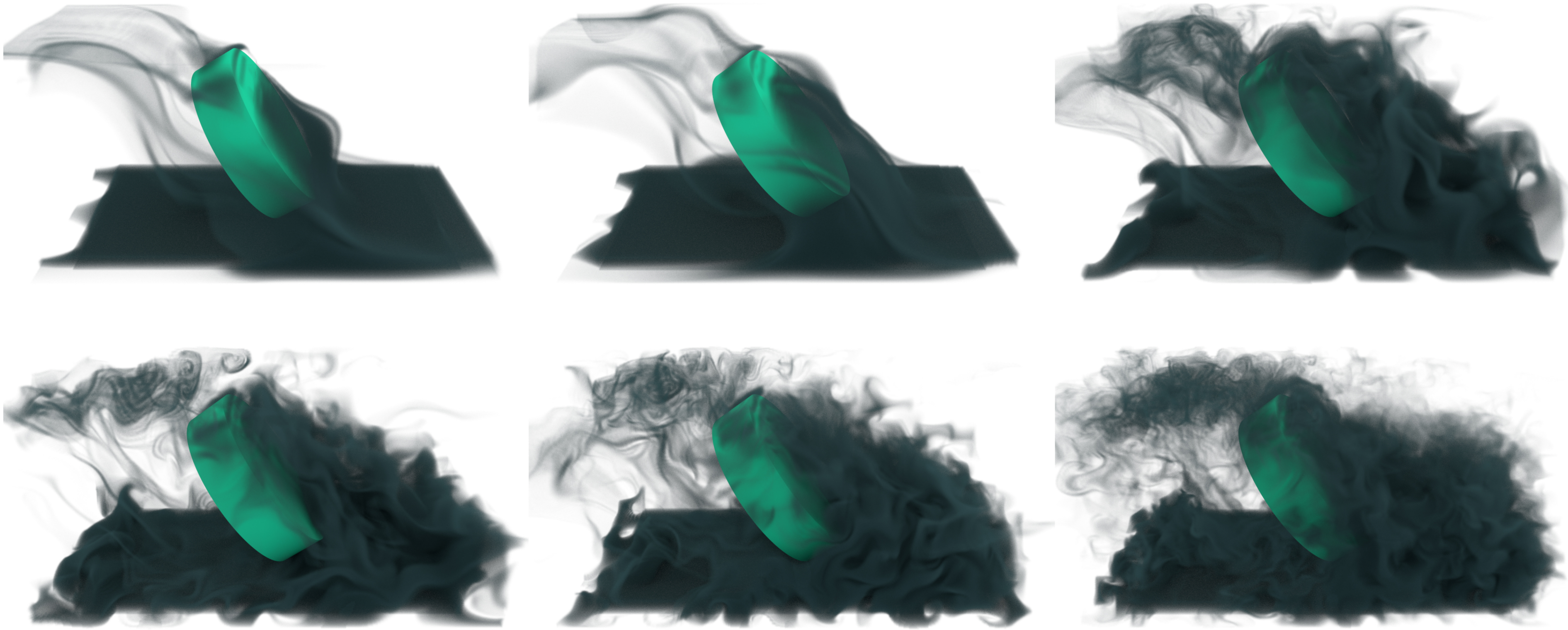
Scalable Laplacian Eigenfluids



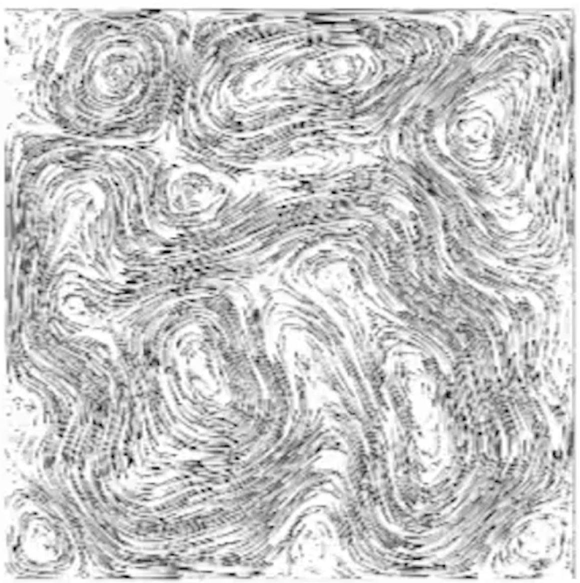
Qiaodong Cui Pradeep Sen Theodore Kim
University of California, Santa Barbara Pixar

Fluid Simulation using Laplacian Eigenfunctions

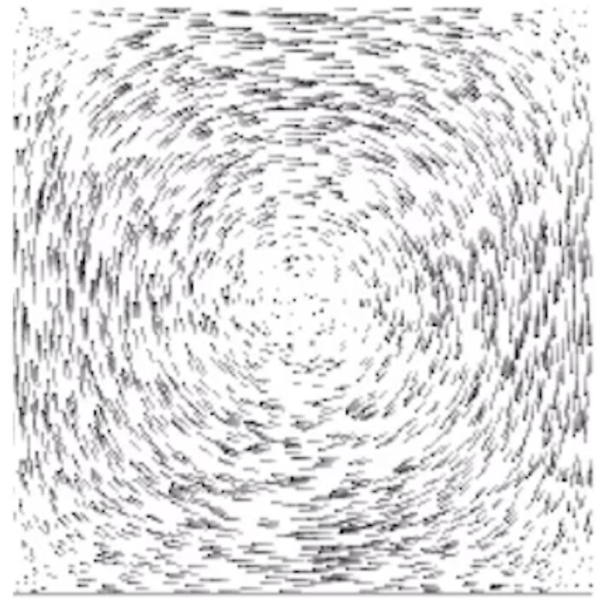
Eigenfluids

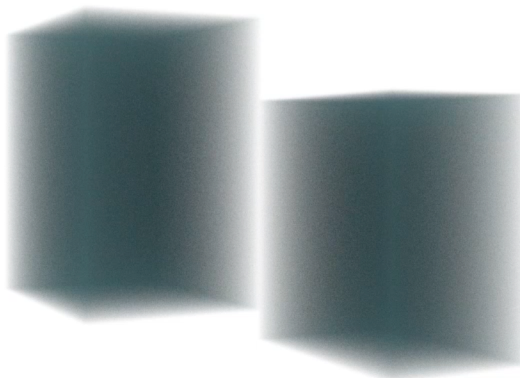
[DeWitt et al. 2012]

Inviscid



Pressure free



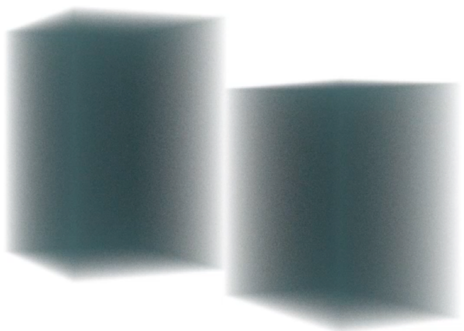


200 eigenfunctions [DeWitt et al. 2012]

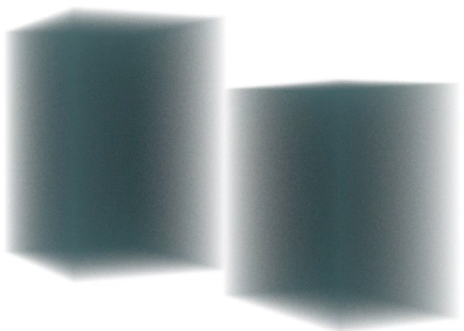
memory: 52 GB
time: 17.2 secs/frame

24K eigenfunctions:

[DeWitt et al. 2012]	memory: 6.1 TB	time: 1.84 hrs/frame
Ours	memory: 26 GB	time: 13 secs/frame



200 eigenfunctions



24K eigenfunctions

Outline

- Previous work
- Laplacian Eigenfluids
- Our methods
- Results
- Conclusions and future work

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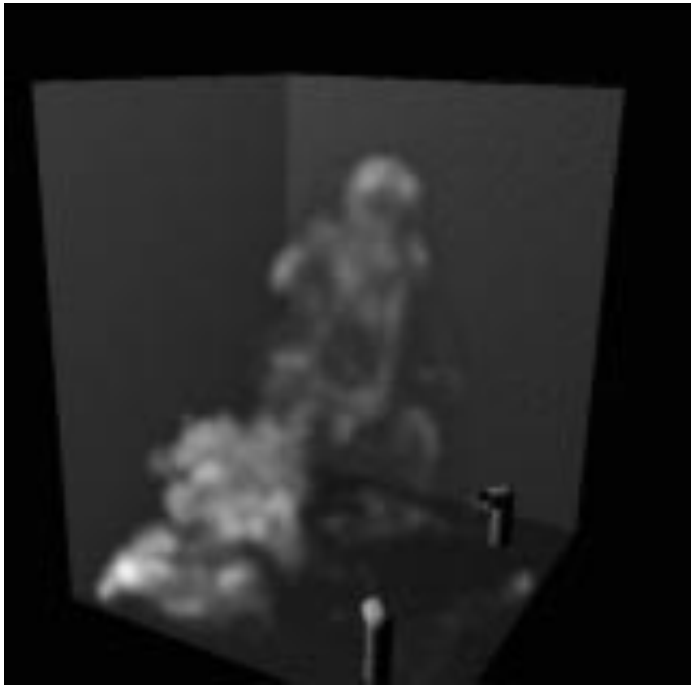
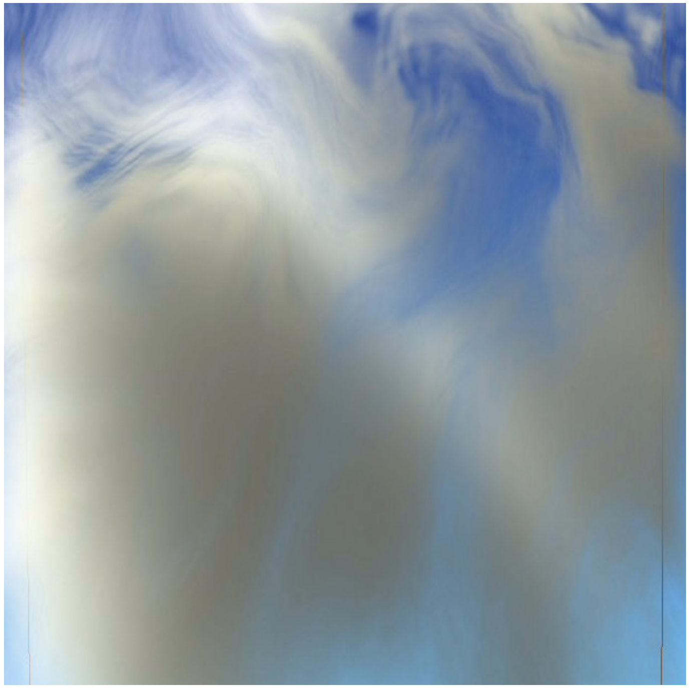
Fluid Simulation

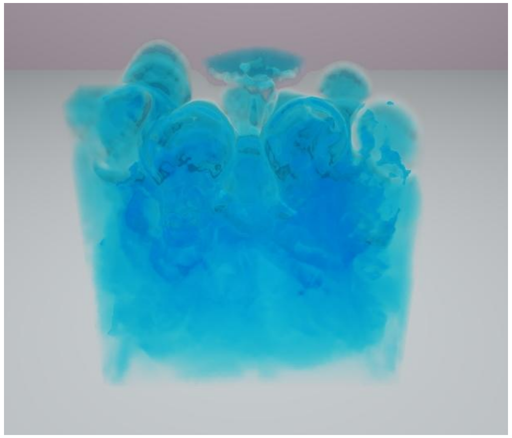


[Foster and Metaxas 1997]

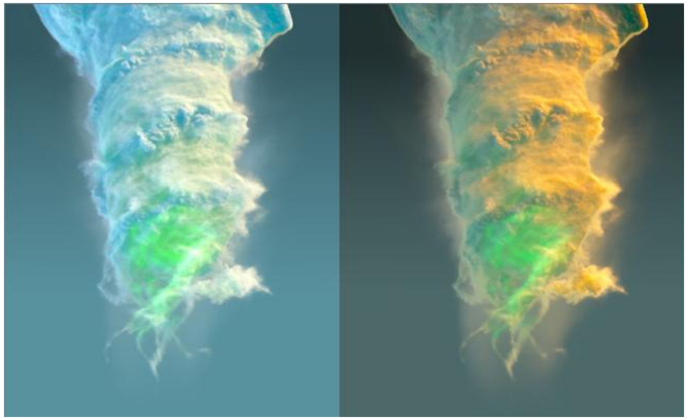


[Stam 1999]

Spectral Solvers



[Long and Reinhard 2009]



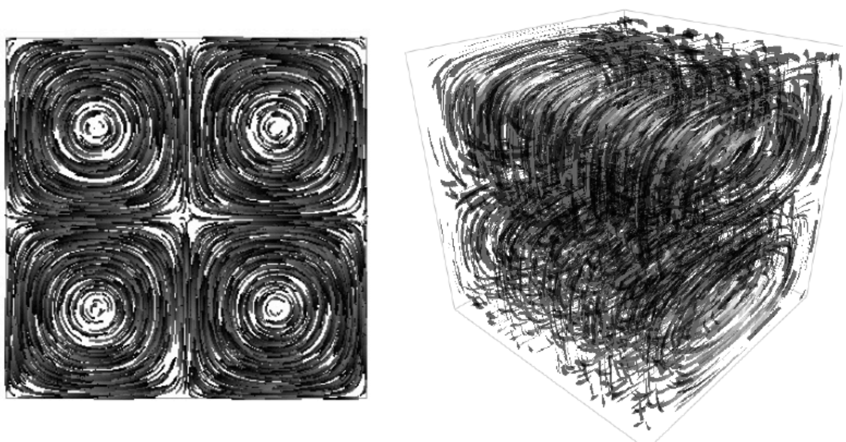
[Henderson 2012]

Inviscid Methods

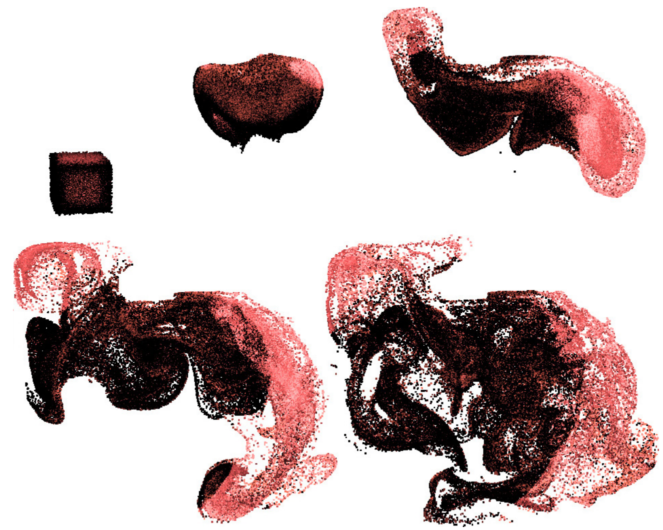
[Mullen et al. 2009]

[Chern et al. 2016]

Laplacian Eigenfluids



[De Witt et al. 2012]



[Liu et al. 2015]

Outline

- Previous work
- Laplacian Eigenfluids
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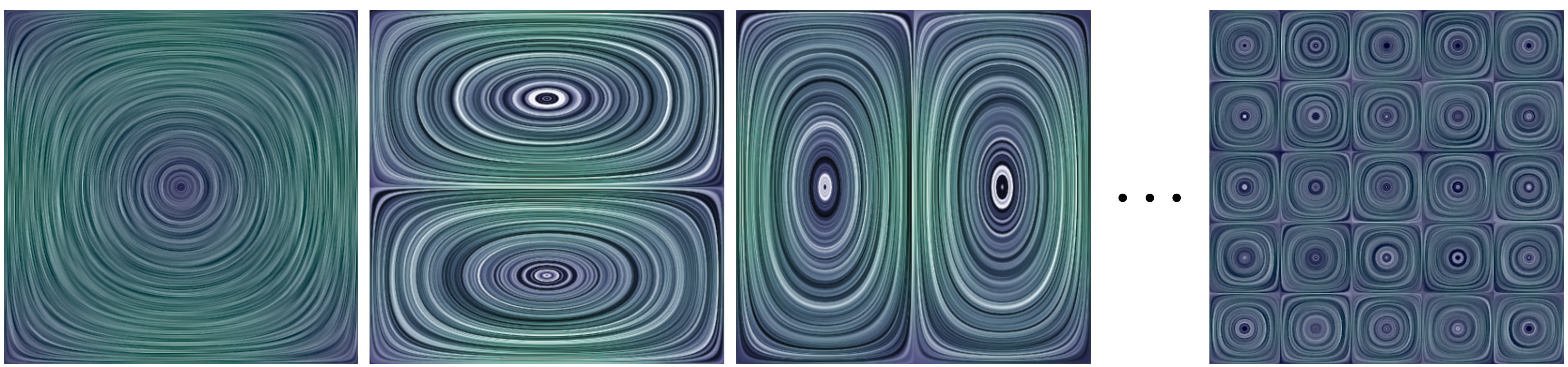
Laplacian Eigenfluids

$$\dot{\mathbf{u}} = -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u} = \sum_{i=1}^{r} w_i \boldsymbol{\Psi_i} = w_1 \boldsymbol{\Psi_1} + w_2 \boldsymbol{\Psi_2} + w_3 \boldsymbol{\Psi_3} + \ \dots \ + w_r \boldsymbol{\Psi_r}$$

$$\nabla \cdot \boldsymbol{\Psi}_i = 0$$

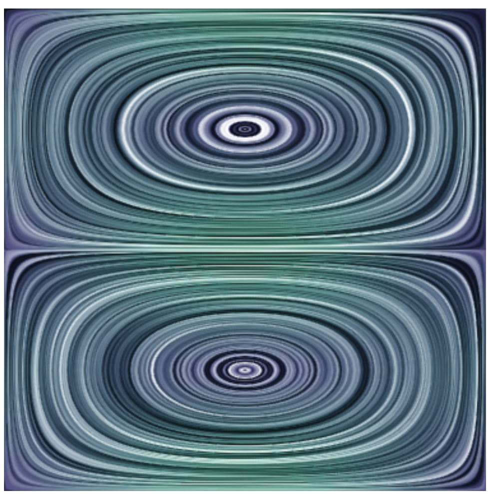


$\boldsymbol{\Psi_1}$ $\boldsymbol{\Psi_2}$ $\boldsymbol{\Psi_3}$ $\dots$ $\boldsymbol{\Psi_r}$

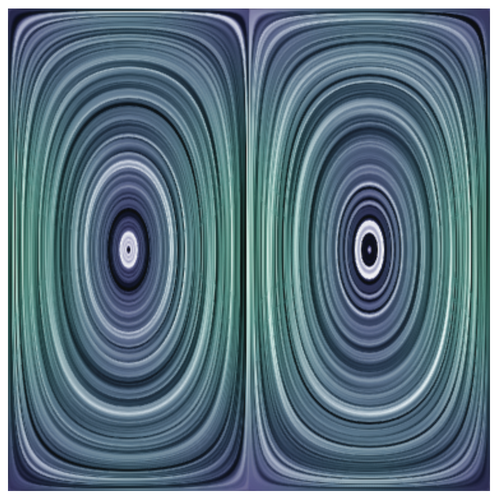
Laplacian Eigenfunctions

$$\mathbf{\Psi}_x(\mathbf{x},\mathbf{k}) = -\frac{1}{|\mathbf{k}|}\sin(\mathbf{k}_x x)\cos(\mathbf{k}_y y)$$

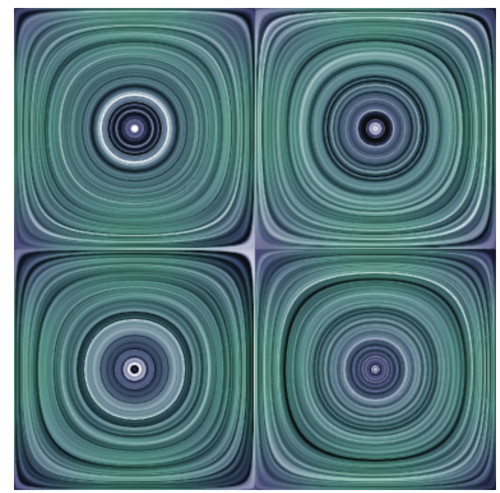
$$\mathbf{\Psi}_y(\mathbf{x},\mathbf{k}) = \frac{1}{|\mathbf{k}|}\cos(\mathbf{k}_x x)\sin(\mathbf{k}_y y)$$



$\mathbf{k}_x = 1,\ \mathbf{k}_y = 2$

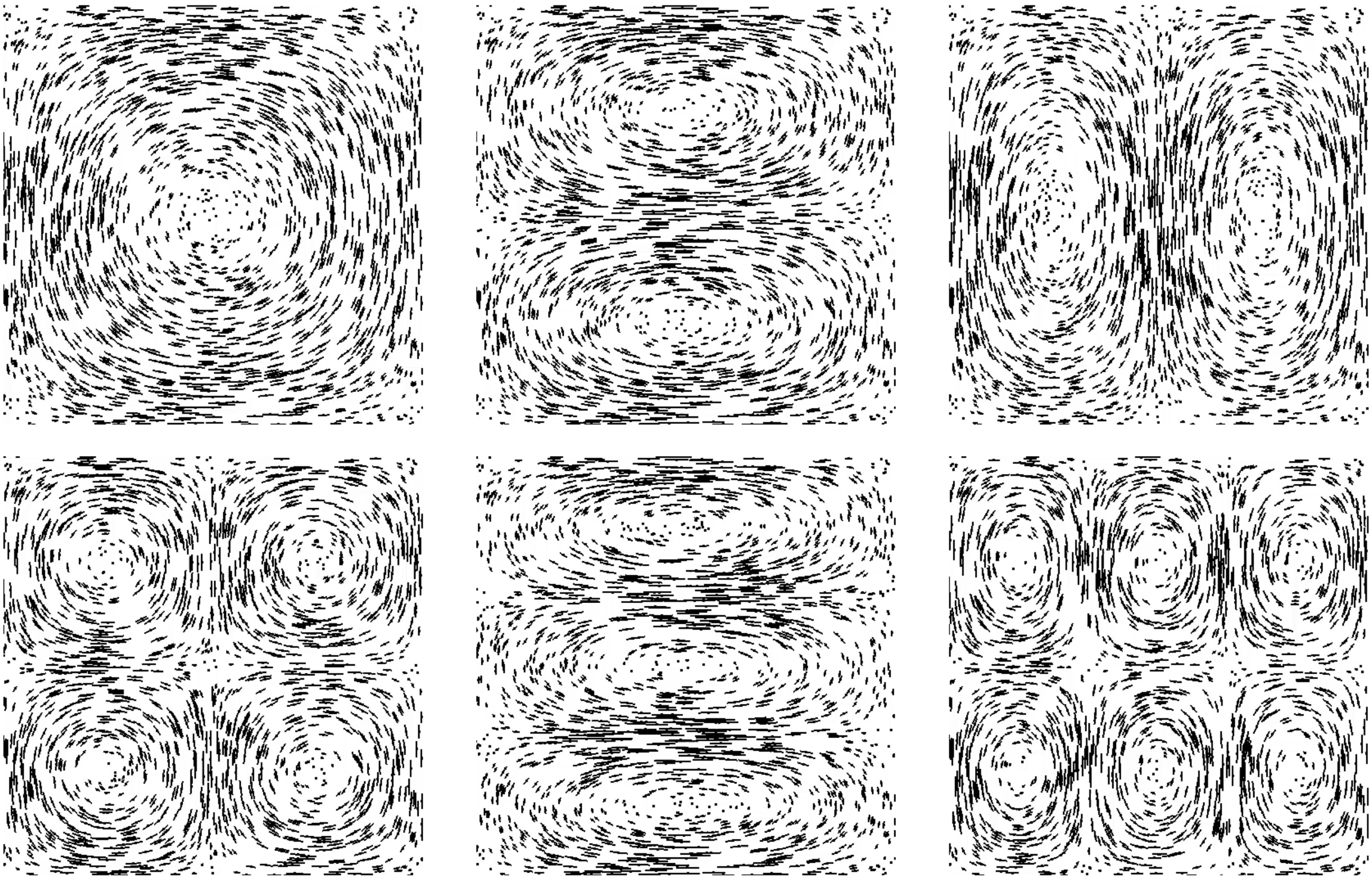


$\mathbf{k}_x = 2,\ \mathbf{k}_y = 1$

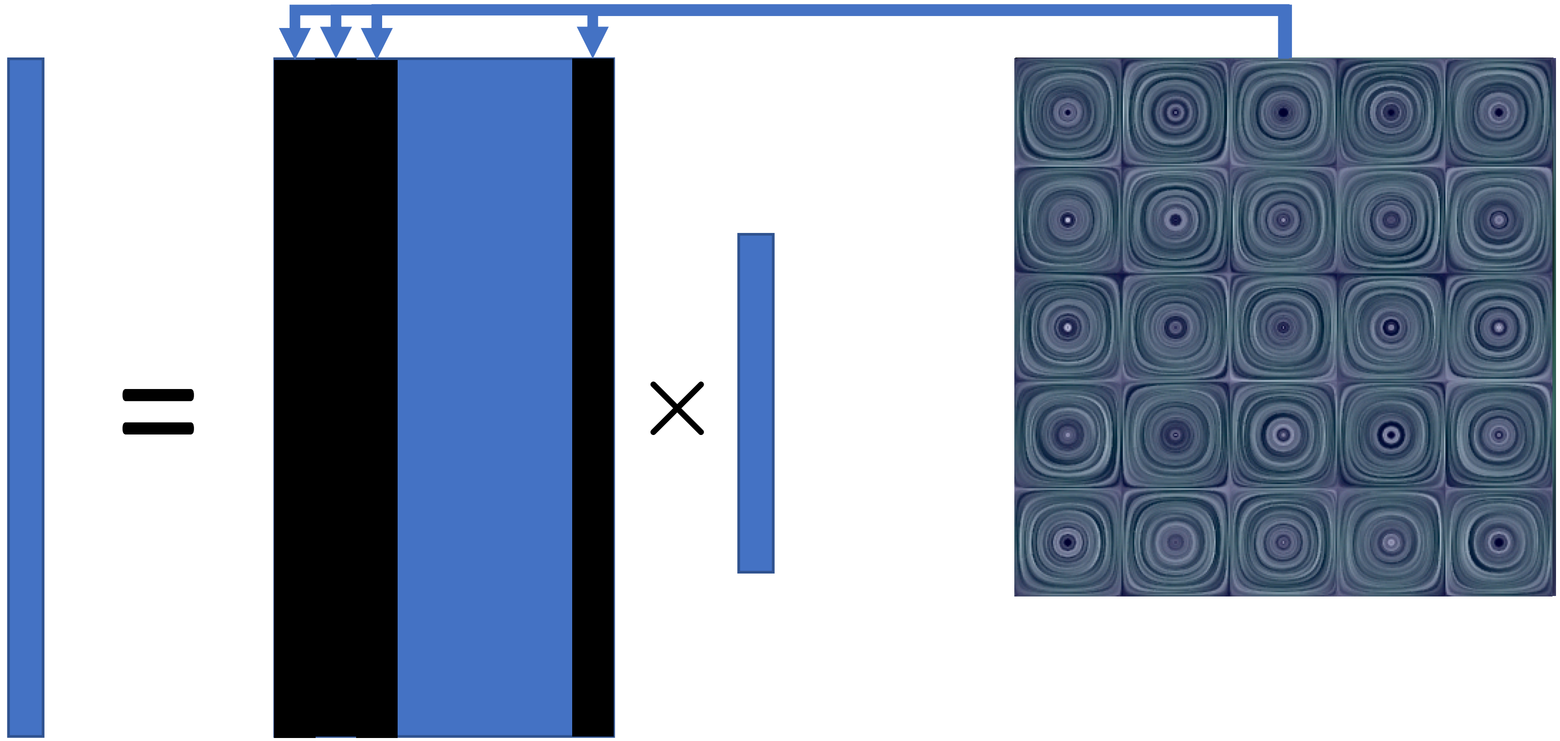


$\mathbf{k}_x = 2,\ \mathbf{k}_y = 2$

Laplacian Eigenfunctions



Reconstruction Bottleneck



$\mathbf{u} \in \mathbb{R}^{N^3}$ $\mathbf{U} \in \mathbb{R}^{N^3 \times r}$ $\mathbf{W} \in \mathbb{R}^{r}$ $r \approx 1000$ 50.5 GB

$\mathbf{U} = \{\mathbf{\Psi}_1, \mathbf{\Psi}_2, ..., \mathbf{\Psi}_r\}$ 84%

Analytical Basis

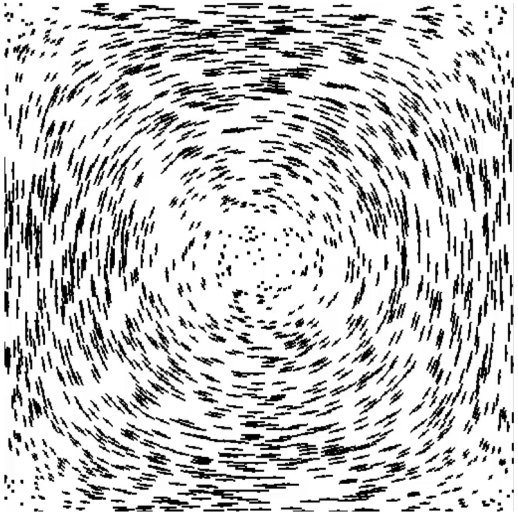
- Basis take analytical forms under rectangular domain

$$\mathbf{\Psi}_x(\mathbf{x},\mathbf{k}) = -\frac{1}{|\mathbf{k}|}\sin(\mathbf{k}_x x)\cos(\mathbf{k}_y y)$$
$$\mathbf{\Psi}_y(\mathbf{x},\mathbf{k}) = \frac{1}{|\mathbf{k}|}\cos(\mathbf{k}_x x)\sin(\mathbf{k}_y y)$$

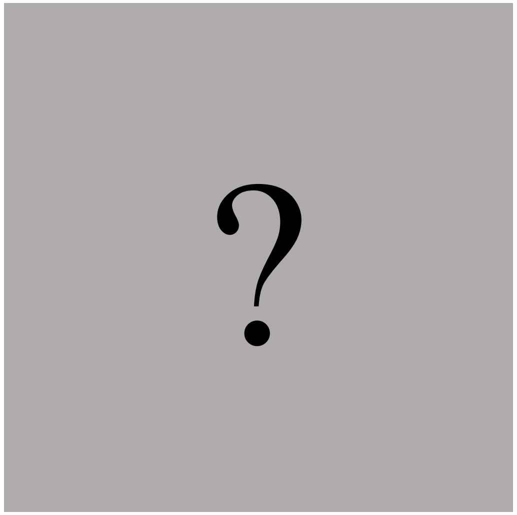
- Removes basis storage
- 5-7 times slower
- $r \approx 1000$ 44.1 seconds vs 9.5 seconds

Boundary Conditions

Dirichlet:



Neumann:



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  - Analytical basis functions with DCT
  - Dynamics
  - Other features
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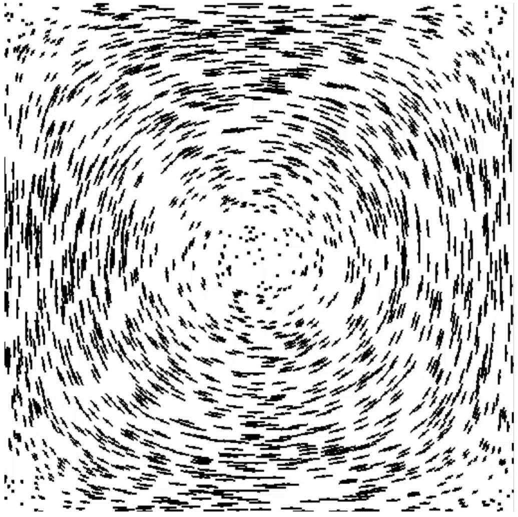
Analytical Basis with DCT

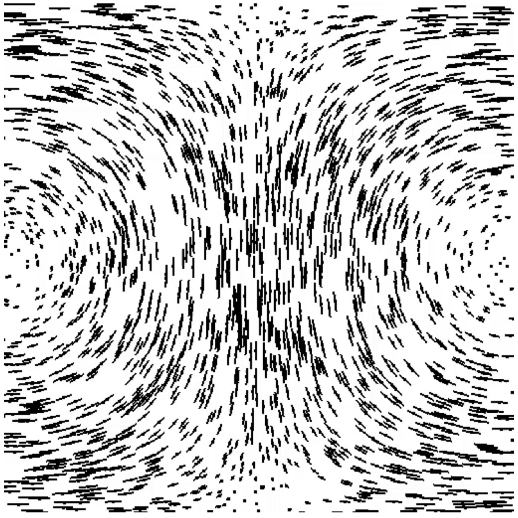
- Memory complexity: $O(rN^3) \longrightarrow O(r)$
- Time complexity: $O(rN^3) \longrightarrow O(N^3 \log(N))$

Analytical Basis with DCT

Dirichlet:



Neumann:



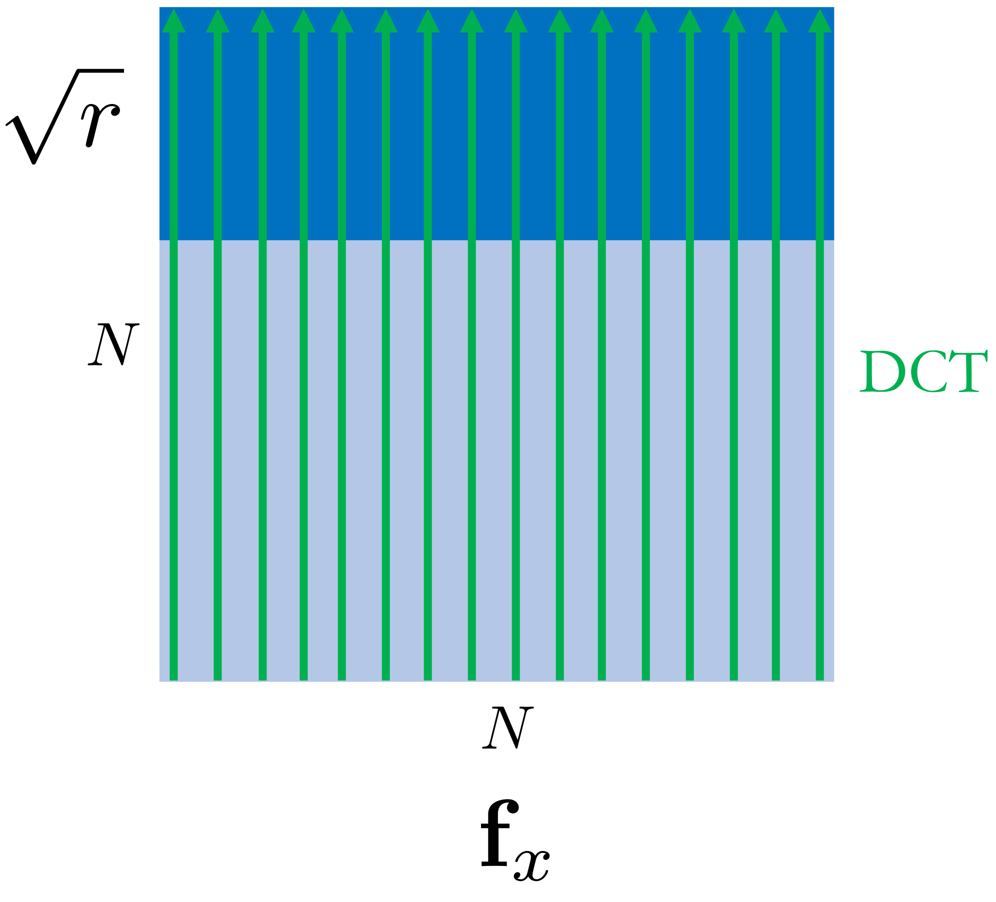
Basis Transformations with DCT

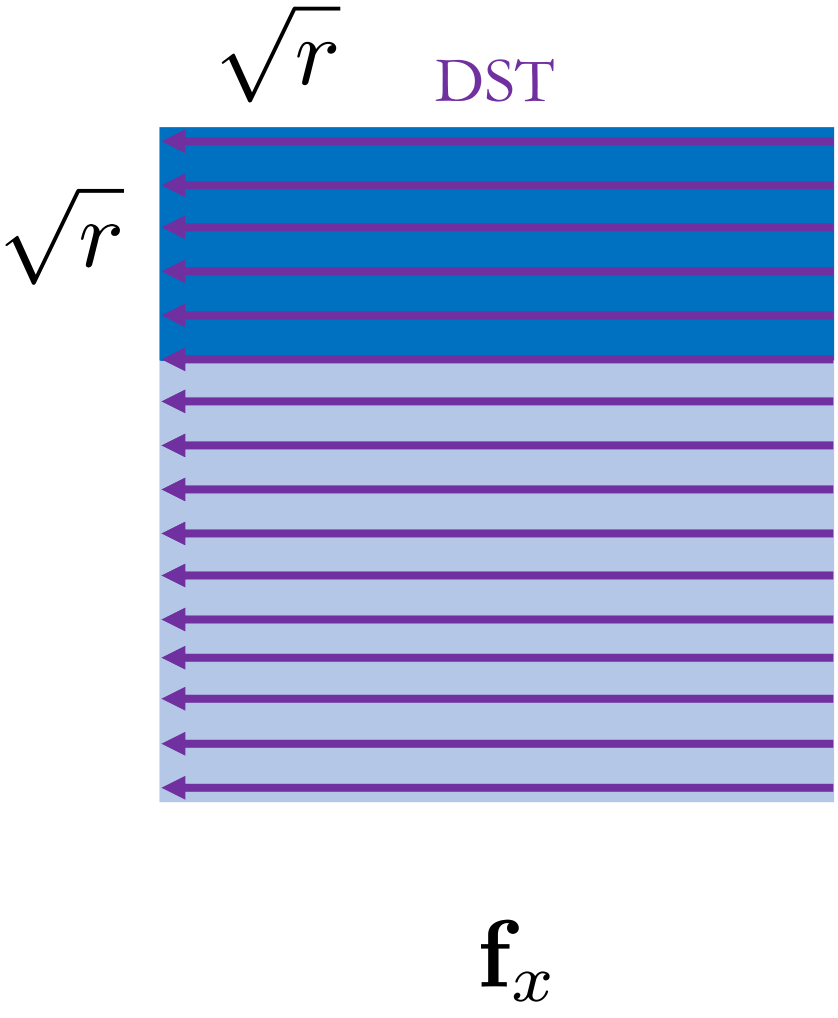
$$\mathbf{\Psi}_x(\mathbf{x}, \mathbf{k}) = -\frac{1}{|\mathbf{k}|} \sin(\mathbf{k}_x x) \cos(\mathbf{k}_y y)$$

$$\mathbf{\Psi}_y(\mathbf{x}, \mathbf{k}) = \frac{1}{|\mathbf{k}|} \cos(\mathbf{k}_x x) \sin(\mathbf{k}_y y)$$

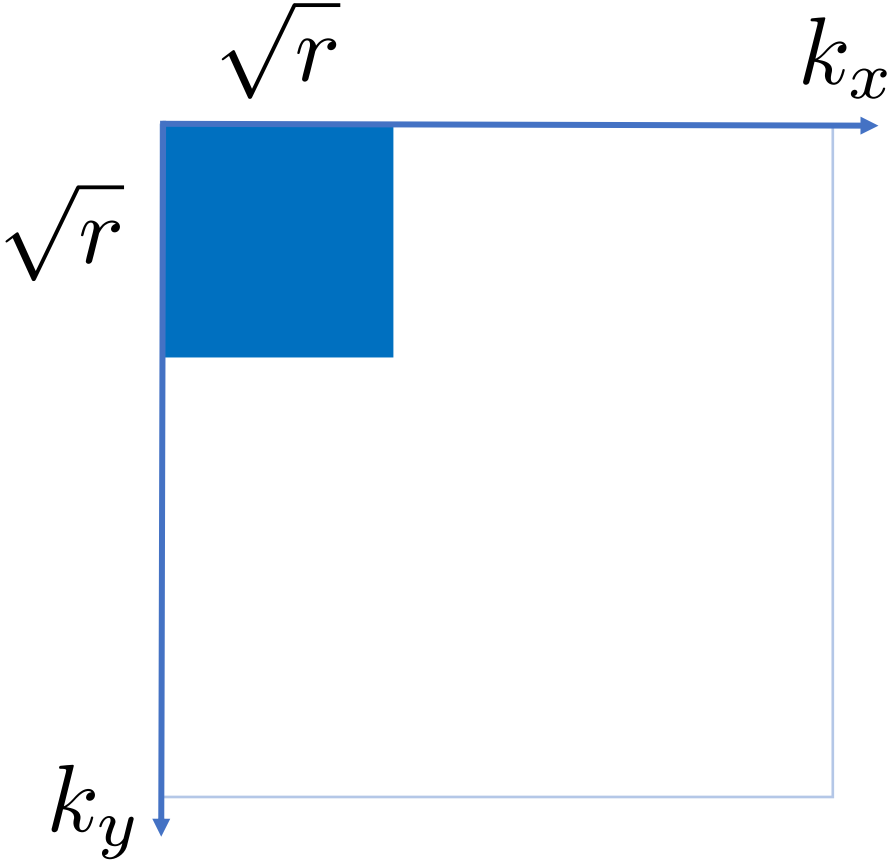
$$\hat{\mathbf{f}}_k = \mathbf{f} \cdot \mathbf{\Psi}_k = \langle \mathbf{f}_x, \mathbf{\Psi}_x(\mathbf{k}) \rangle + \langle \mathbf{f}_y, \mathbf{\Psi}_y(\mathbf{k}) \rangle$$

$$\langle \mathbf{f}_x, \mathbf{\Psi}_x(\mathbf{k}) \rangle = -\frac{1}{|\mathbf{k}|} k_y \boxed{\int_\Omega \mathbf{f}_x \sin(k_x x) \cos(k_y y) dx dy}$$


$\sqrt{r}$
N
DCT
N
$\mathbf{f}_x$



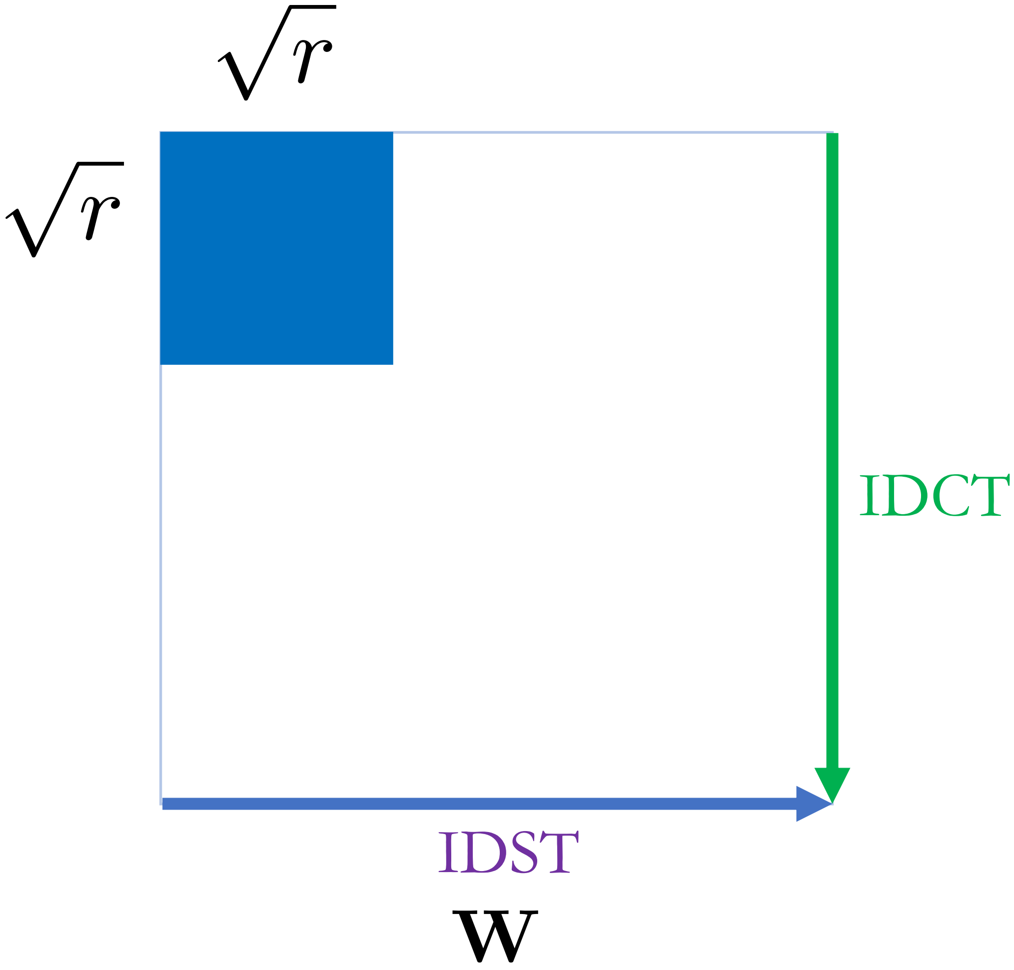
$\sqrt{r}$
DST
$\sqrt{r}$
$\mathbf{f}_x$



$$\langle \mathbf{f}_x, \mathbf{\Psi}_x(\mathbf{k}) \rangle$$

Velocity reconstruction

$$\mathbf{\Psi}_x(\mathbf{x},\mathbf{k}) = -\frac{1}{|\mathbf{k}|}\sin(\mathbf{k}_x x)\cos(\mathbf{k}_y y)$$



Analytical Basis with DCT

- Memory complexity: $O(rN^3) \rightarrow O(r)$ 226×
- Time complexity: $O(rN^3) \rightarrow O(N^3 \log(N))$ 95×

128^3 Grid, 1000 basis functions

Dirichlet

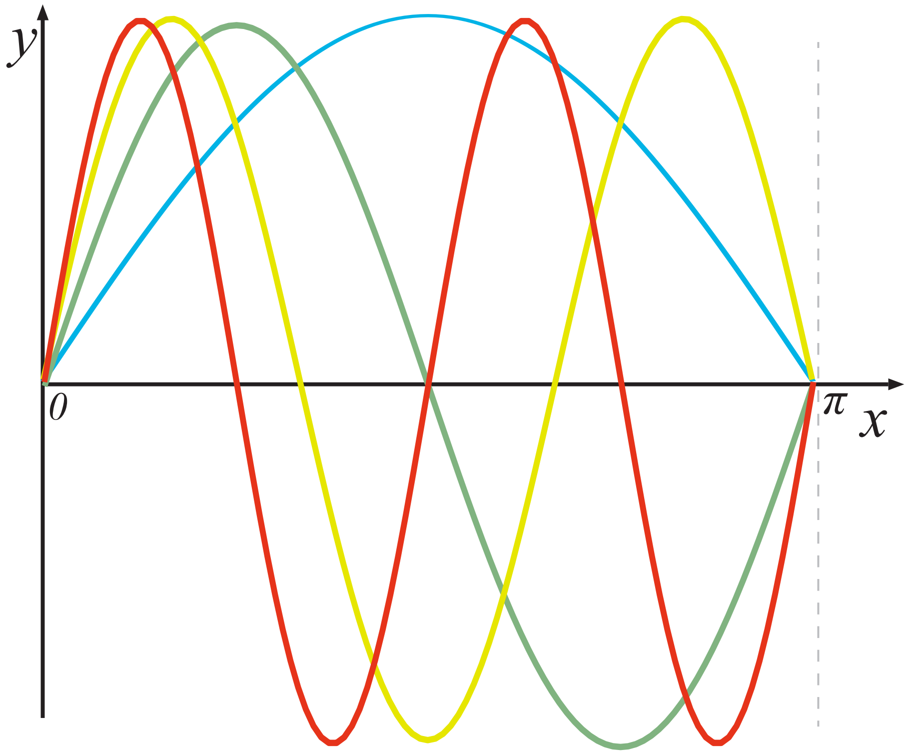
[De Witt et al. 2012]

$$\mathbf{\Psi}_x\big|_{x=0,\pi} = 0$$

$$\mathbf{\Psi}_y\big|_{y=0,\pi} = 0$$

$$\mathbf{\Psi}_x(\mathbf{x},\mathbf{k}) = -\frac{1}{|\mathbf{k}|}\sin(\mathbf{k}_x x)\cos(\mathbf{k}_y y)$$

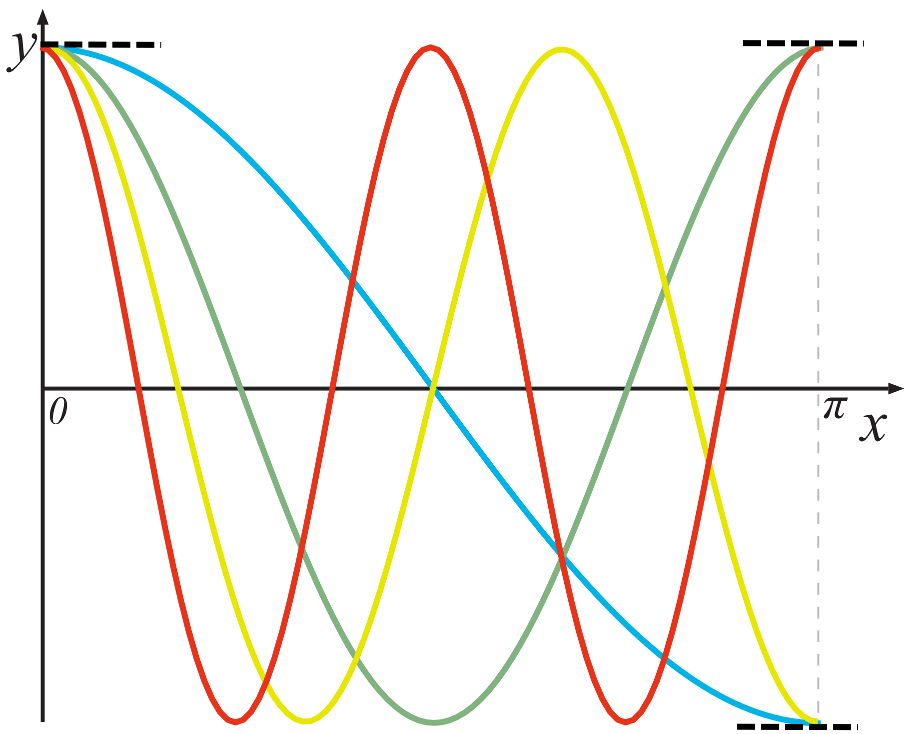
$$\mathbf{\Psi}_y(\mathbf{x},\mathbf{k}) = \frac{1}{|\mathbf{k}|}\cos(\mathbf{k}_x x)\sin(\mathbf{k}_y y)$$



$$y = \sin(\mathbf{k}_x x)$$

Extension to Neumann

$$\frac{\partial \mathbf{\Psi}_x}{\partial x}\Big|_{x=0,\pi} = 0$$



$$y = \cos(\mathbf{k}_x x)$$

Extension to Neumann

$$\frac{\partial \boldsymbol{\Psi}_x}{\partial x}|_{x=0,\pi} = 0 \qquad \boldsymbol{\Psi}_y\big|_{y=0,\pi} = 0$$

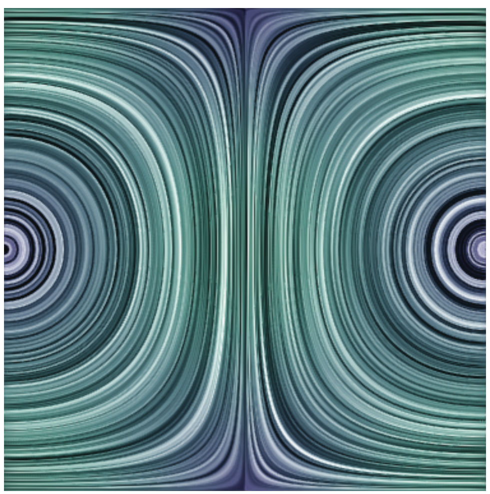
$$\begin{cases} \nabla^2 \boldsymbol{\Psi}(\mathbf{x}, \mathbf{k}) = -|\mathbf{k}|^2 \boldsymbol{\Psi}(\mathbf{x}, \mathbf{k}) \\ \nabla \cdot \boldsymbol{\Psi}(\mathbf{x}, \mathbf{k}) = 0 \end{cases}$$

$$\boldsymbol{\Psi}_x(\mathbf{x}, \mathbf{k}) = \tfrac{1}{|\mathbf{k}|} \cos(\mathbf{k}_x x) \cos(\mathbf{k}_y y)$$
$$\boldsymbol{\Psi}_y(\mathbf{x}, \mathbf{k}) = \tfrac{1}{|\mathbf{k}|} \sin(\mathbf{k}_x x) \sin(\mathbf{k}_y y)$$

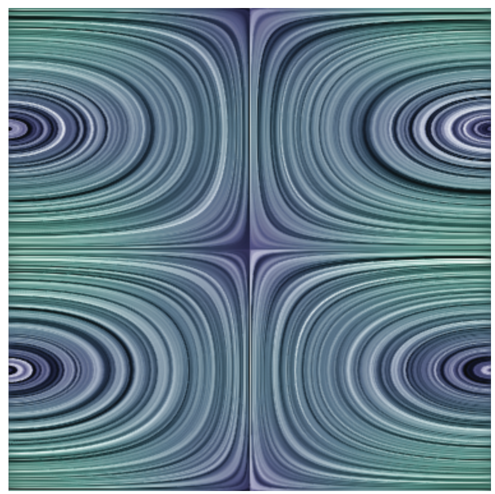
Extension to Neumann

$$\mathbf{\Psi}_x(\mathbf{x}, \mathbf{k}) = \frac{1}{|\mathbf{k}|} \cos(\mathbf{k}_x x) \cos(\mathbf{k}_y y)$$

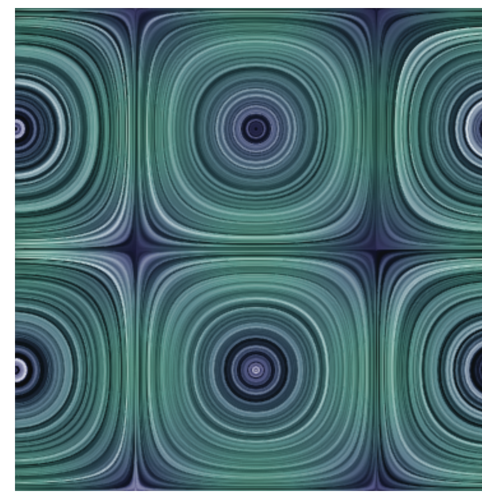
$$\mathbf{\Psi}_y(\mathbf{x}, \mathbf{k}) = \frac{1}{|\mathbf{k}|} \sin(\mathbf{k}_x x) \sin(\mathbf{k}_y y)$$



$\mathbf{k}_x = 1,\ \mathbf{k}_y = 1$

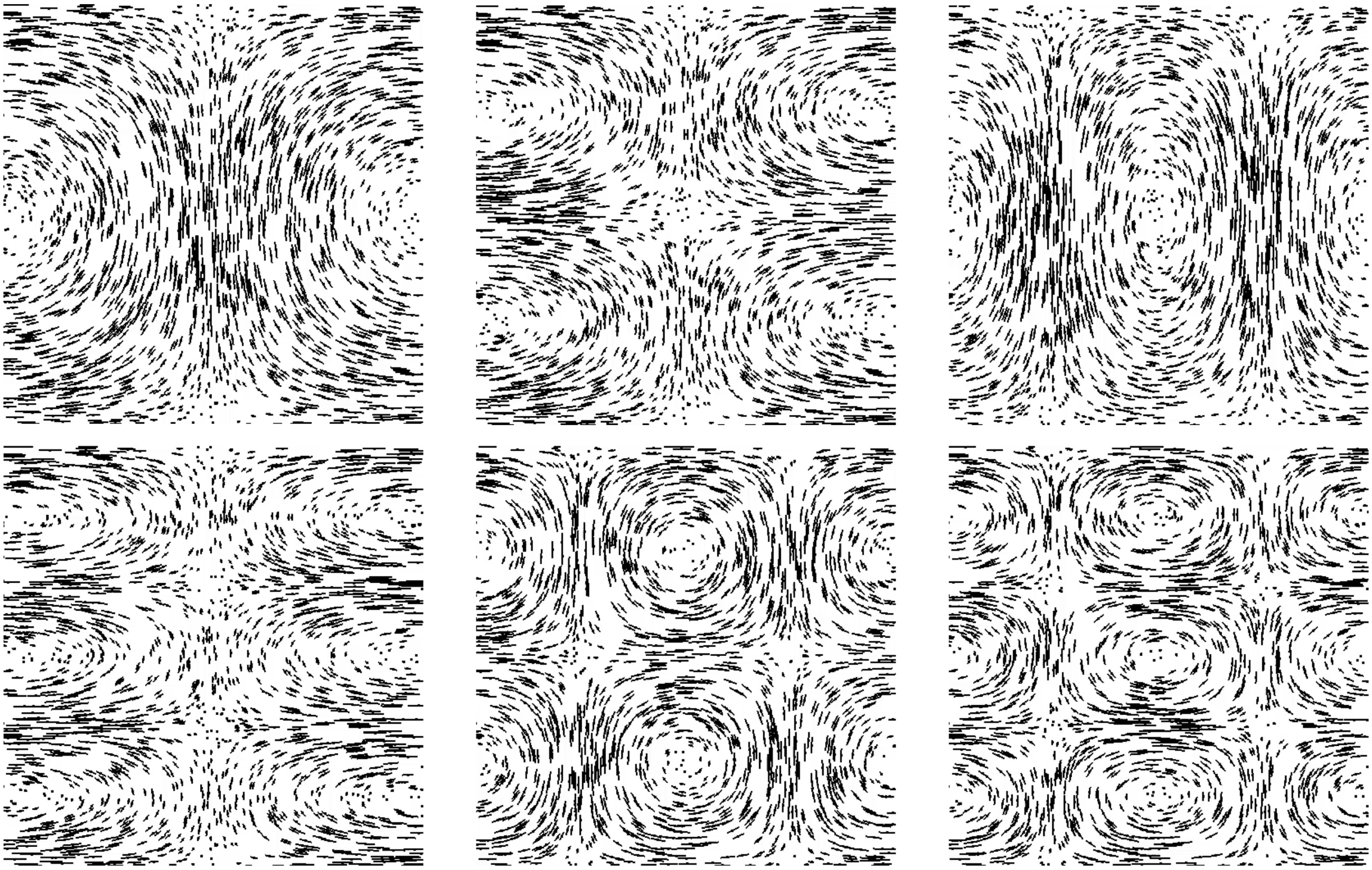


$\mathbf{k}_x = 1,\ \mathbf{k}_y = 2$



$\mathbf{k}_x = 2,\ \mathbf{k}_y = 2$

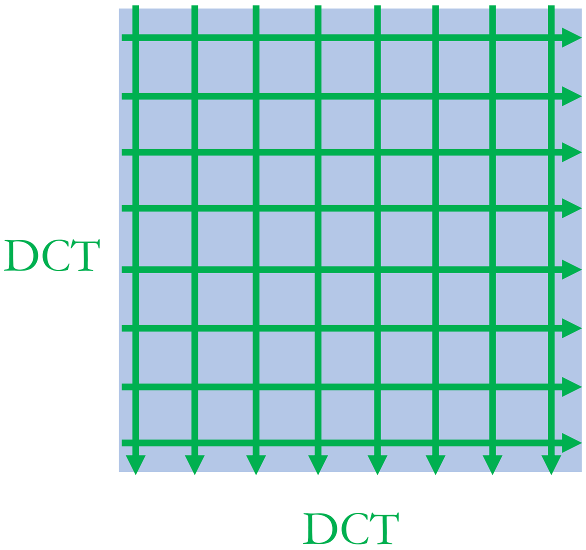
Extension to Neumann



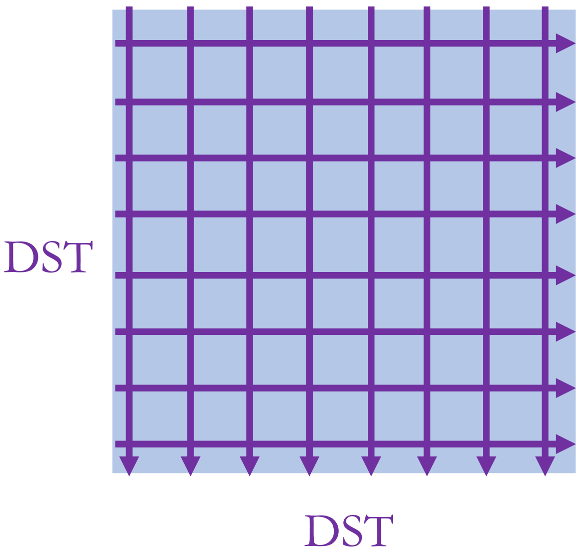
Extension to Neumann

$$\mathbf{\Psi}_x(\mathbf{x},\mathbf{k}) = \frac{1}{|\mathbf{k}|}\cos(\mathbf{k}_x x)\cos(\mathbf{k}_y y)$$

$$\mathbf{\Psi}_y(\mathbf{x},\mathbf{k}) = \frac{1}{|\mathbf{k}|}\sin(\mathbf{k}_x x)\sin(\mathbf{k}_y y)$$



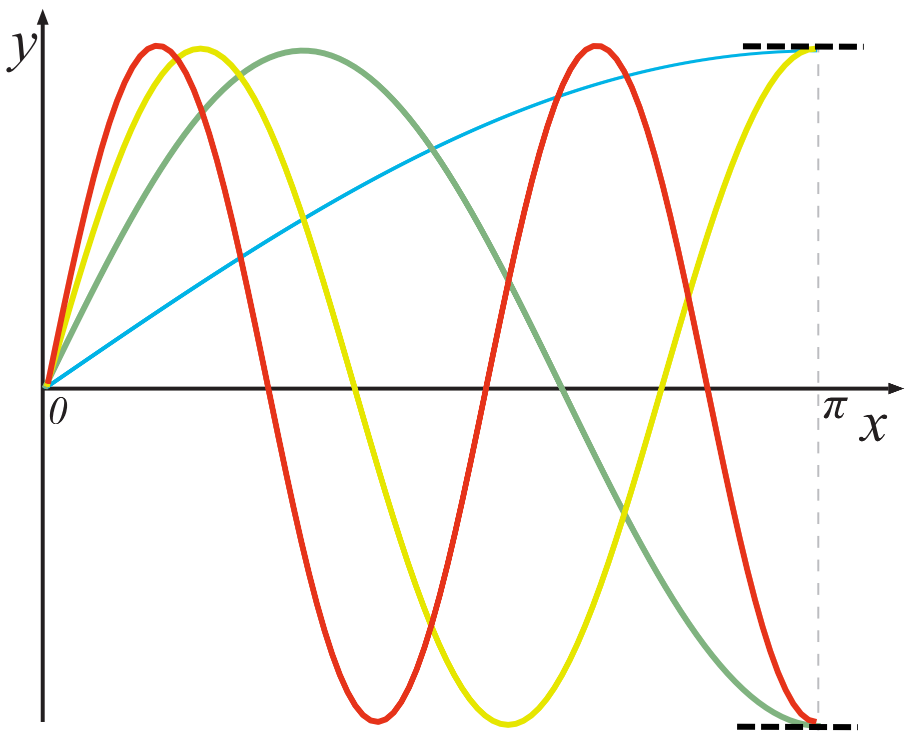
$\mathbf{\Psi}_x$



$\mathbf{\Psi}_y$

Mixed Boundary Conditions

$$\mathbf{\Psi}_x\big|_{x=0} = 0 \qquad \frac{\partial \mathbf{\Psi}_x}{\partial x}\Big|_{x=\pi} = 0$$



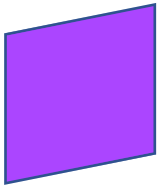
$$y = \sin((\mathbf{k}_x - 0.5)x), \ \ \mathbf{k}_x \in \mathbb{Z}^+$$

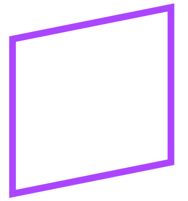
Mixed Boundary Conditions

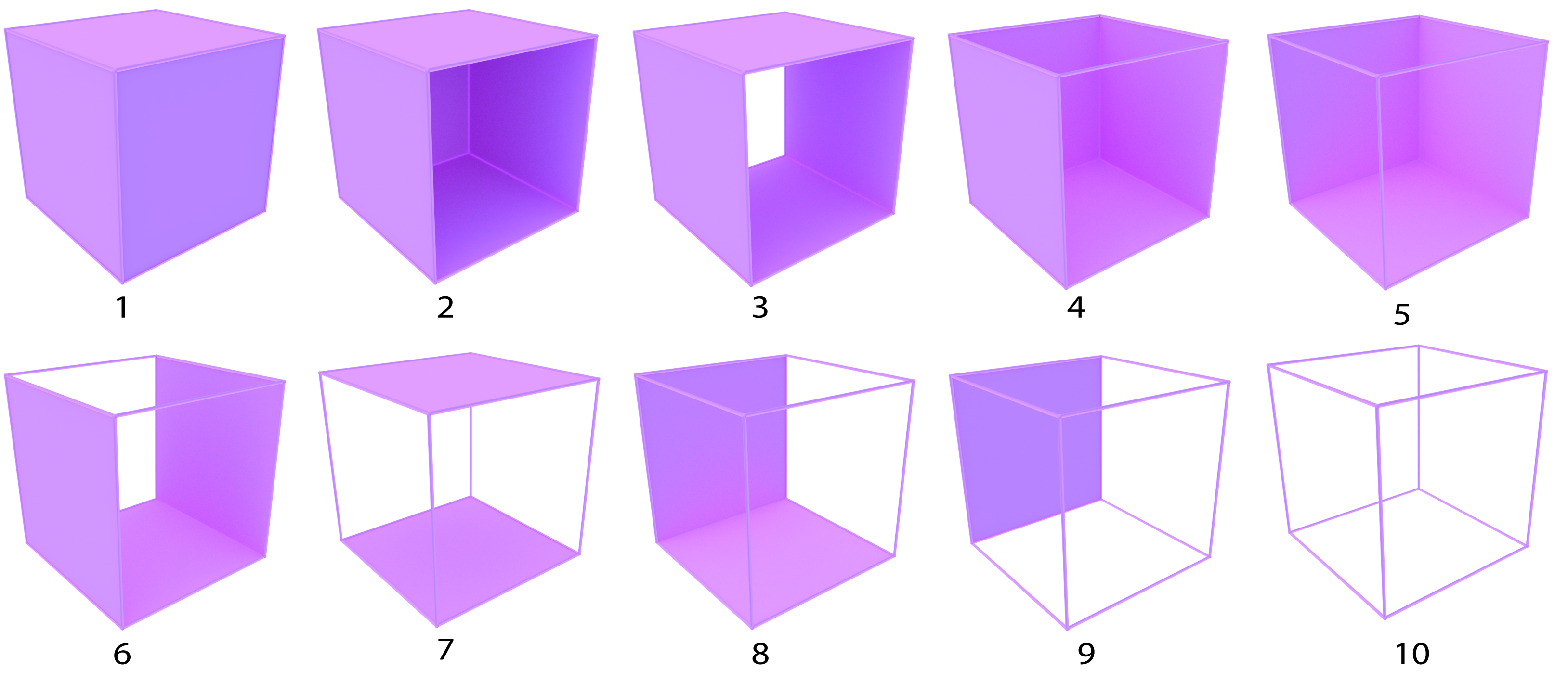
$$\mathbf{\Psi}_x|_{x=0} = 0 \qquad \frac{\partial \mathbf{\Psi}_x}{\partial x}|_{x=\pi} = 0$$

$$\mathbf{\Psi}_x(\mathbf{x}, \mathbf{k}) = -\frac{1}{\mathbf{k}} \sin((\mathbf{k}_x - 0.5)x) \cos(\mathbf{k}_y y)$$
$$\mathbf{\Psi}_y(\mathbf{x}, \mathbf{k}) = \frac{1}{\mathbf{k}} \cos((\mathbf{k}_x - 0.5)x) \sin(\mathbf{k}_y y)$$

3D Basis Functions


Dirichlet


Neumann



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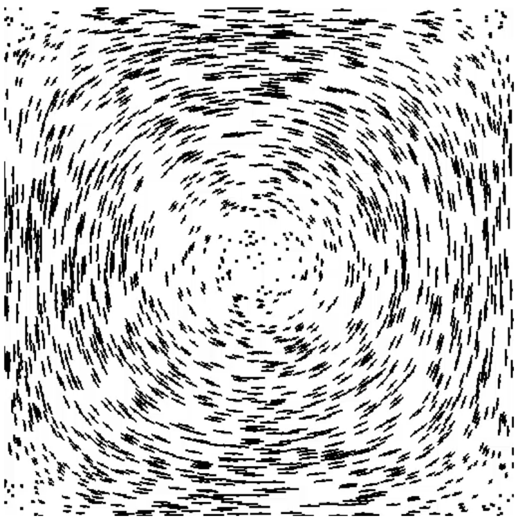
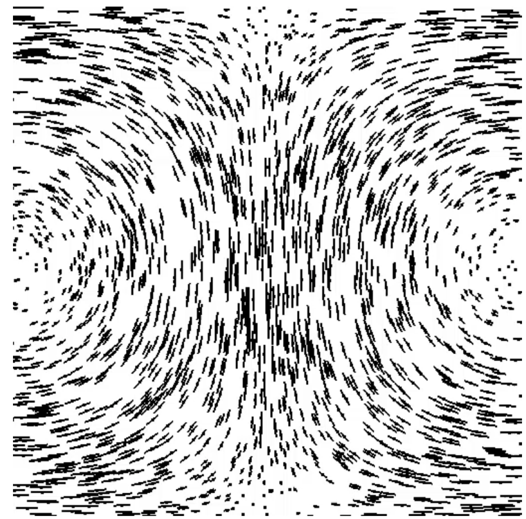
Dynamics

	Dirichlet:	Neumann:
		
Original Eigenfluids		Blows up
Variational		

Dynamics

$$\dot{\mathbf{u}} = -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

Dynamics

$$\mathbf{u} = \sum_{i=1}^{r} w_i \mathbf{\Psi}_i \quad \nabla^2 \mathbf{\Psi}_i = \lambda_i \mathbf{\Psi}_i$$

Diffusion: $\dot{\mathbf{u}} = \nu \nabla^2 \mathbf{u}$

$$w_k^{t+1} = w_k^t e^{\nu \lambda_i \Delta t}$$

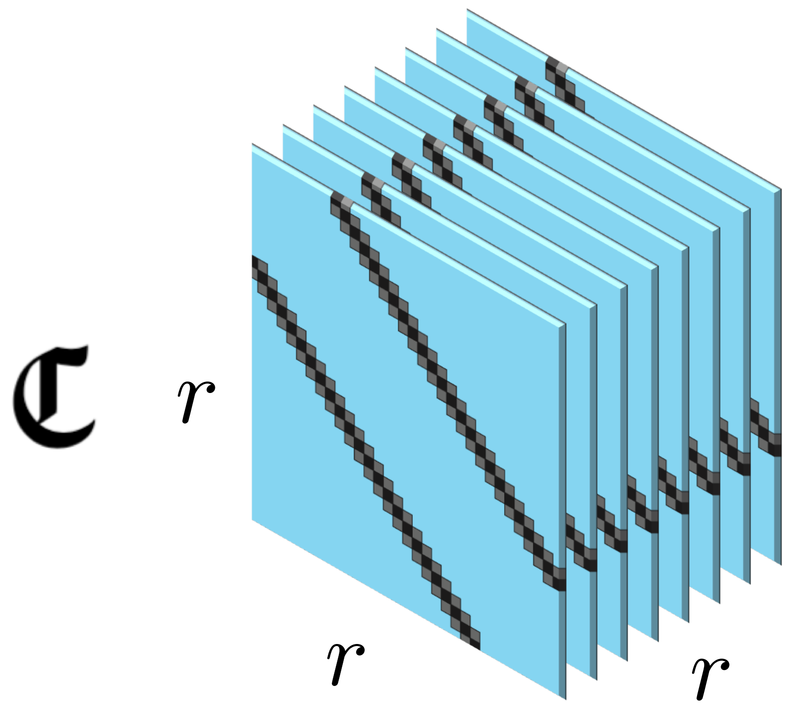
External force: $\dot{\mathbf{u}} = \mathbf{f}$

$$\hat{\mathbf{f}}_k = \mathbf{f} \cdot \mathbf{\Psi}_k \qquad w_k^{t+1} = \Delta t \hat{\mathbf{f}}_k$$

Advection

$$\dot{\mathbf{u}} = -\mathbf{u} \cdot \nabla \mathbf{u} \longrightarrow \dot{\boldsymbol{\omega}} = \nabla \times (\boldsymbol{\omega} \times \mathbf{u})$$

$$\dot{w}_g = \sum_{h=1}^{r} \sum_{i=1}^{r} w_h w_i \mathfrak{C}(g, h, i)$$



Dynamics

$$\dot{w}_g = \sum_{h=1}^{r} \sum_{i=1}^{r} w_h w_i \mathfrak{C}(g, h, i)$$

$$\phi_h = \nabla \times \mathbf{\Psi}_h$$

$$\mathfrak{C}(g, h, i) = [\nabla \times (\phi_h \times \mathbf{\Psi}_i)] \cdot \phi_g$$

[De Witt et al. 2012]

Dirichlet: OK

Neumann: Blows up

Dynamics

$$\mathfrak{C}(g,h,i) = -\mathfrak{C}(h,g,i)$$

$$\mathfrak{C}(g,h,i) = \int_\Omega (\nabla \times \mathbf{\Psi}_i) \cdot (\mathbf{\Psi}_g \times \mathbf{\Psi}_h) d\Omega$$

[Liu et al. 2015]

Dirichlet: OK

Neumann: OK

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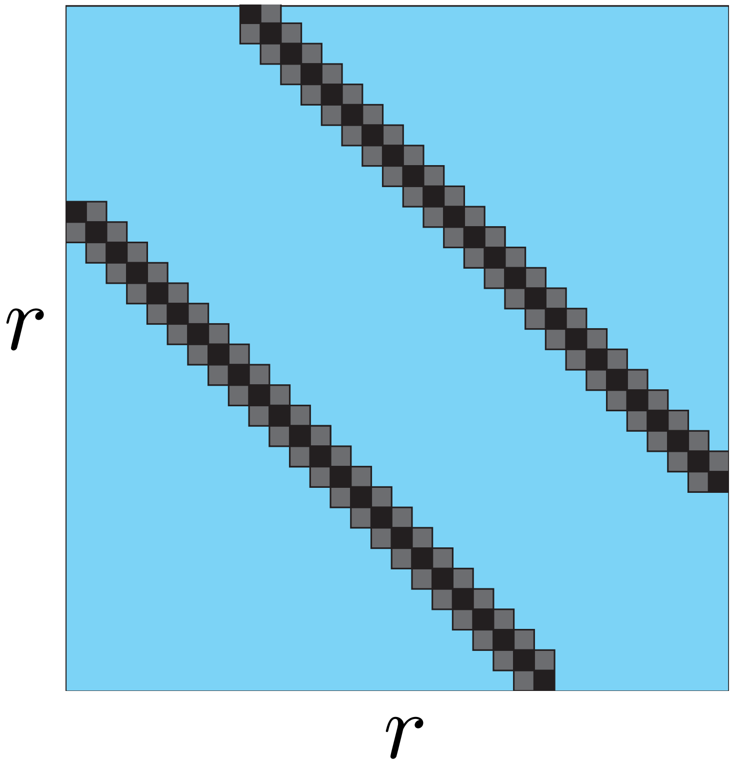
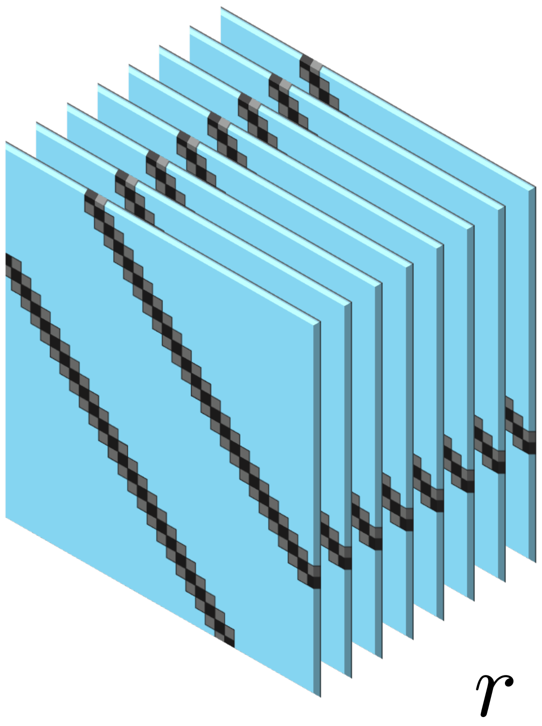
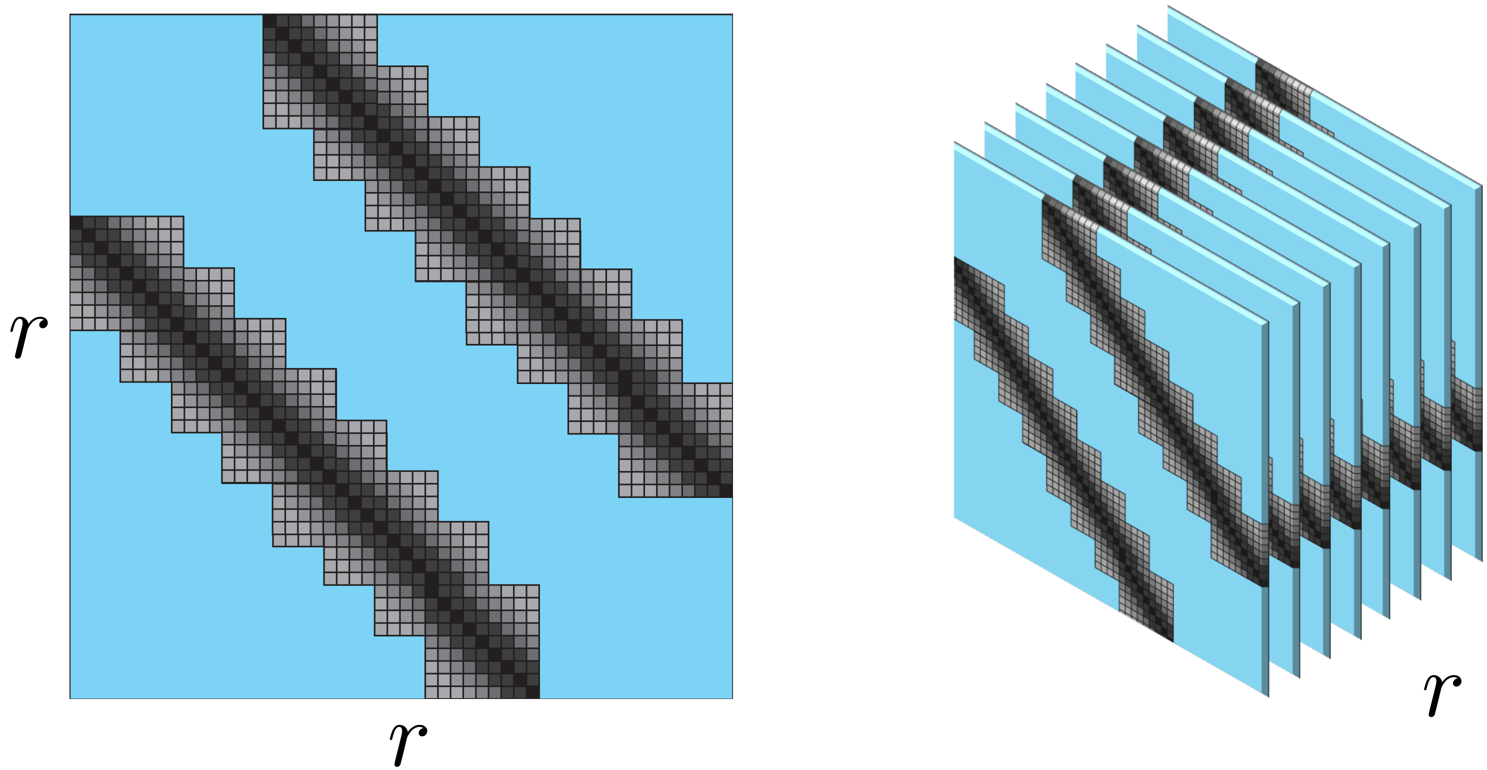
Tensor Compression





Dirichlet: $O(r^2)$

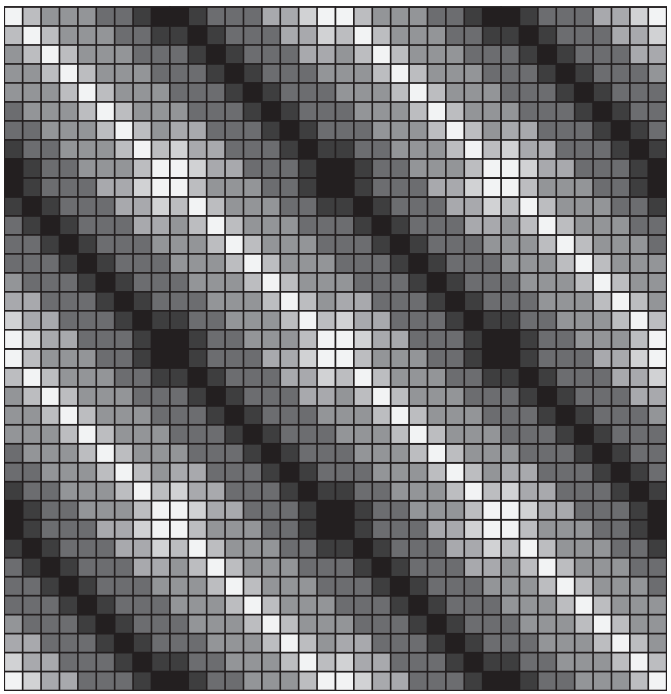
Tensor Compression



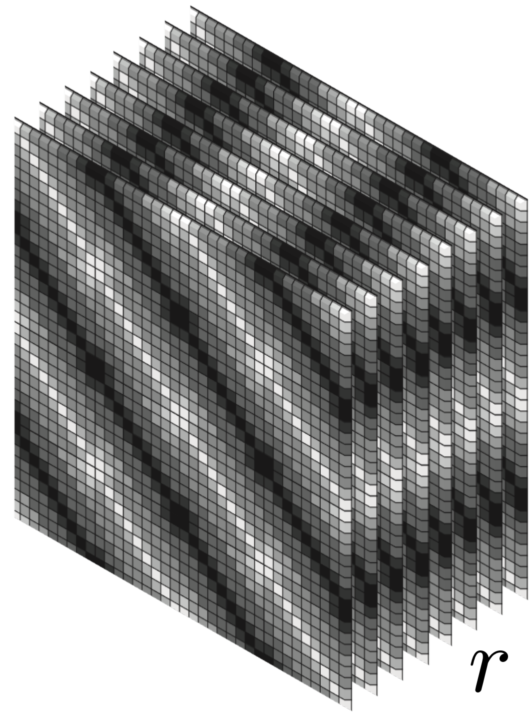
One Neumann direction: $O(r^{2.5})$

Tensor Compression

r



r



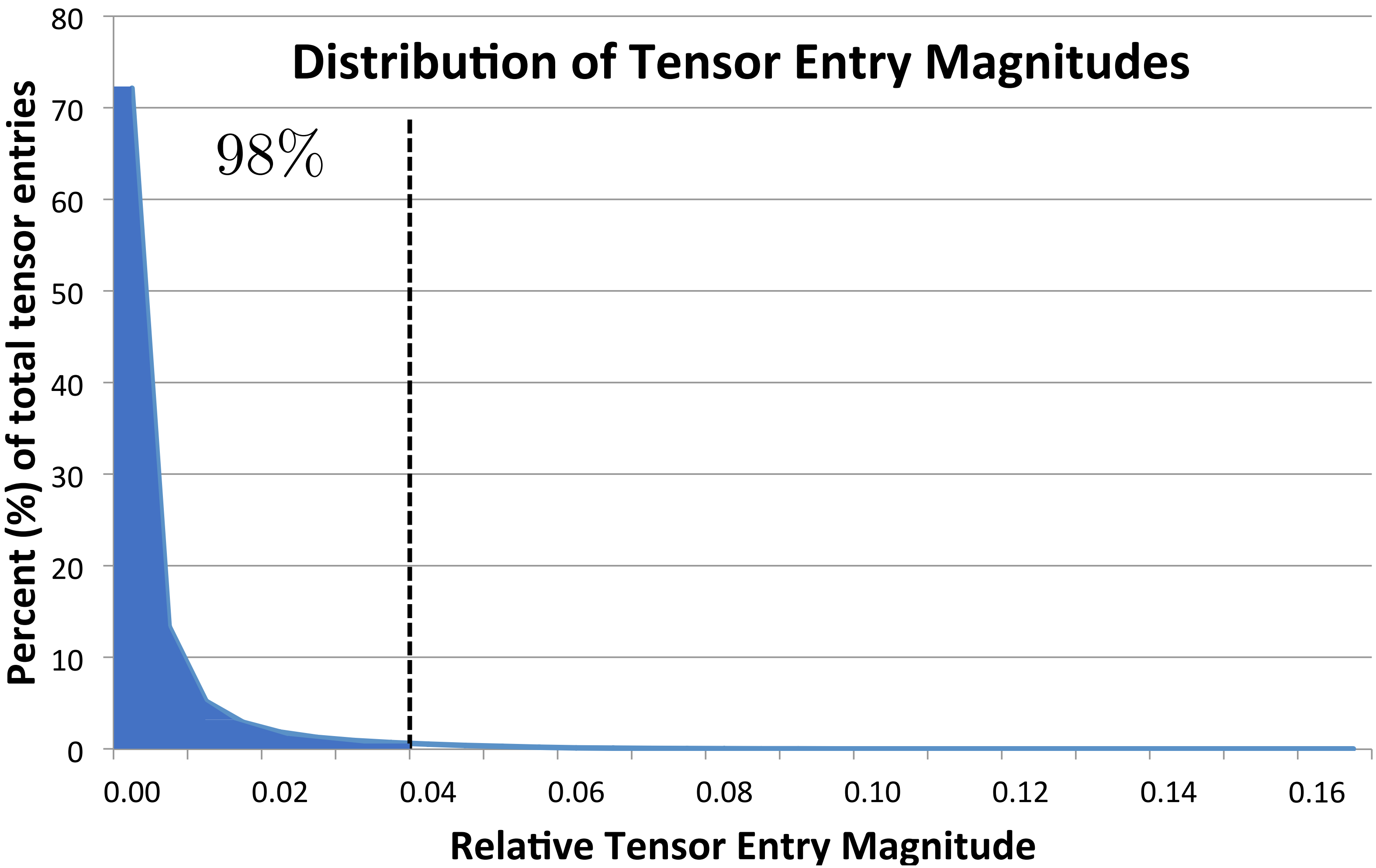
r

Two Neumann directions: $O(r^3)$

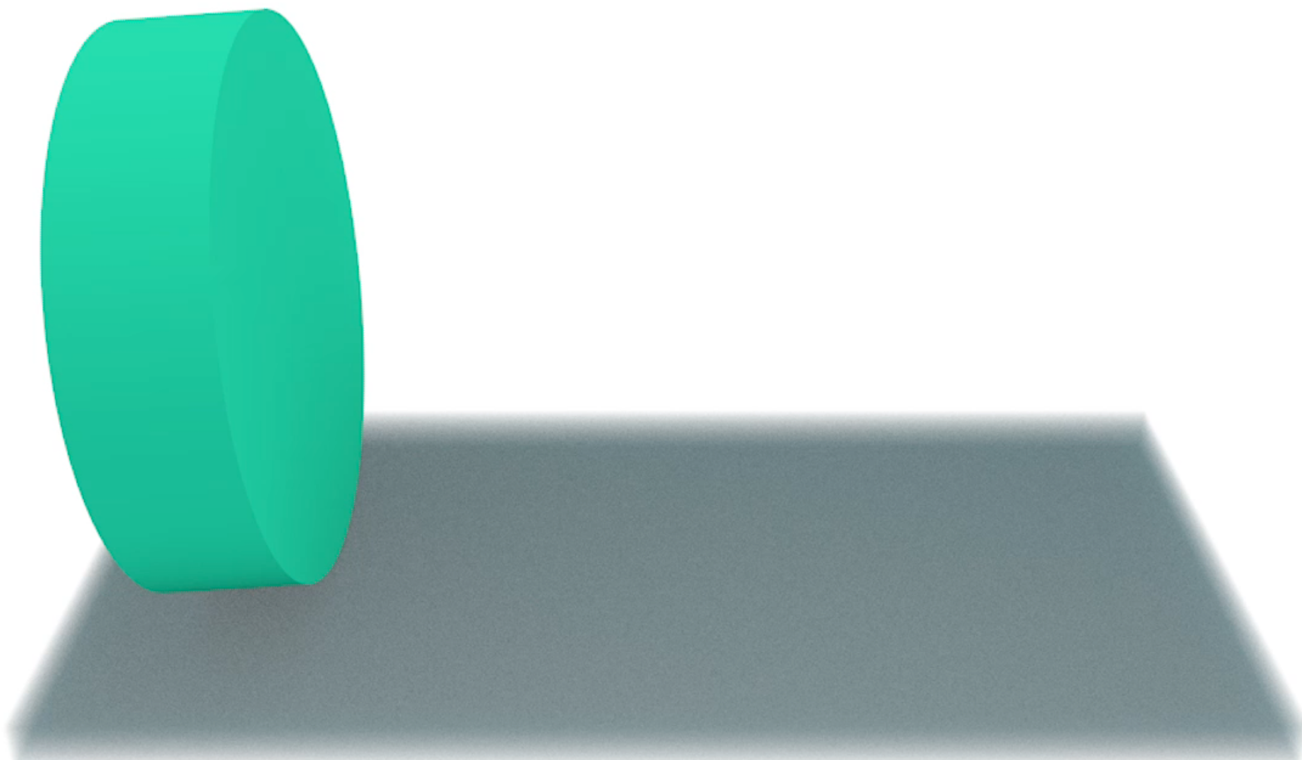
Tensor Sparsity

Dirichlet	$O(r^2)$
One Neumann direction	$O(r^{2+\frac{1}{3}})$
Two Neumann directions	$O(r^{2+\frac{2}{3}})$
Three Neumann directions	$O(r^3)$

Tensor Compression

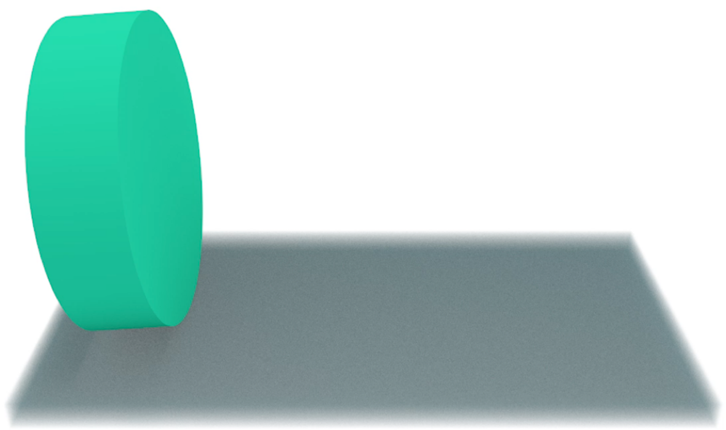


Tensor Compression

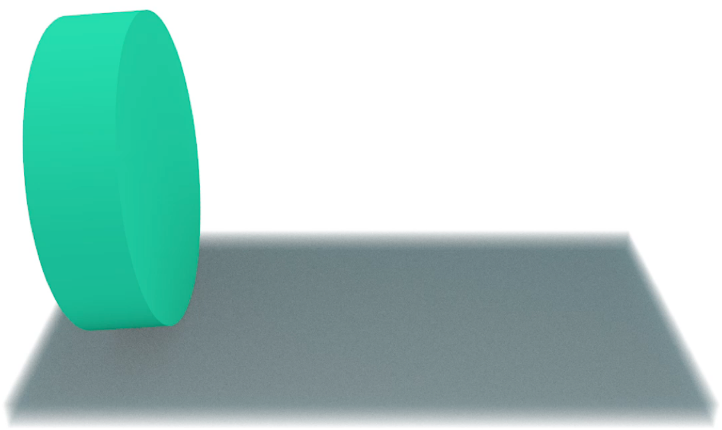


0% discarded, Tensor memory: 26 GB

Tensor Compression

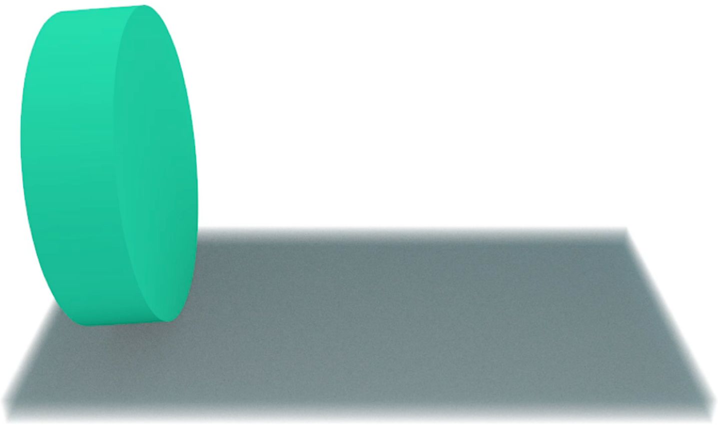


100% discarded, Tensor memory: 0 MB

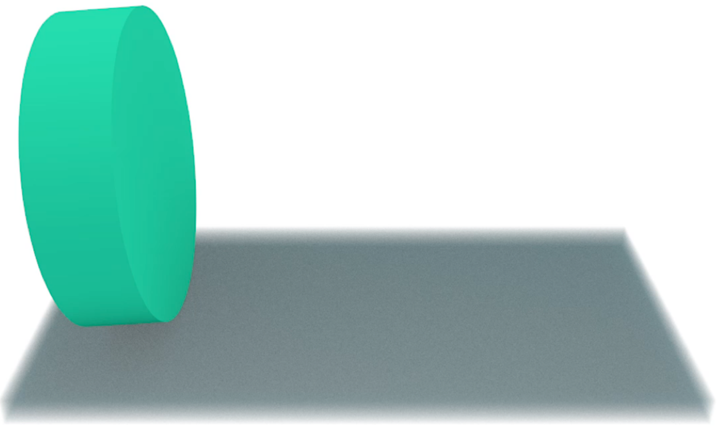


Reference, Tensor memory: 26 GB

Tensor Compression

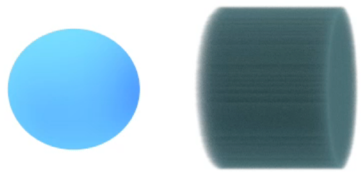


99.9% discarded, Tensor memory: 26 MB

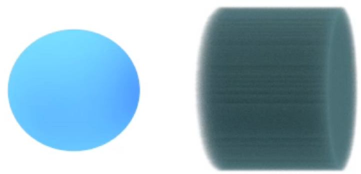


Reference, Tensor memory: 26 GB

Directable Dynamics

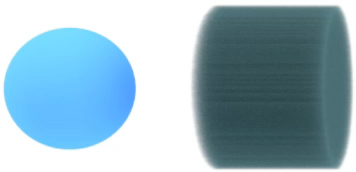


Default tensor

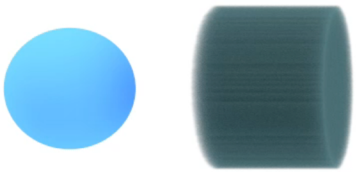


c = 0.0005

Directable Dynamics



Negative weight, c = 0.0005

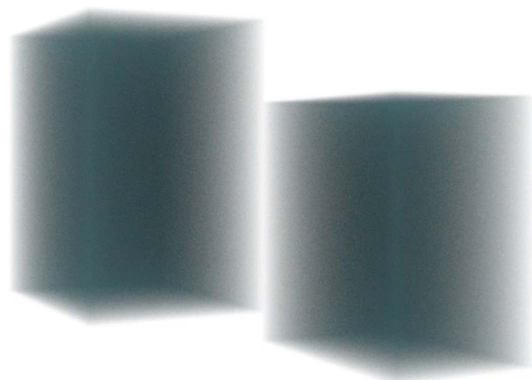


Mixed positive and negative weights

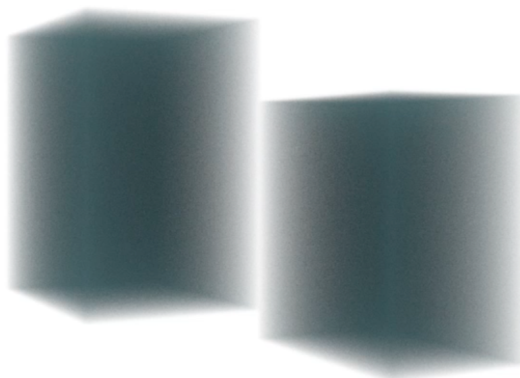
Outline

- Previous work
- Laplacian Eigenfluids
- Our methods
- Results
- Conclusions and future work

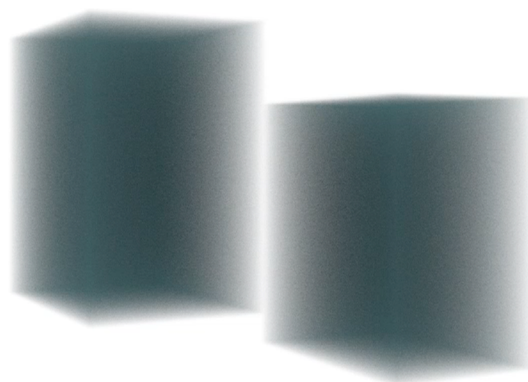
Scaling effects



200 basis functions
[DeWitt et al.] memory: 52 GB
Our Memory: 0.9 GB

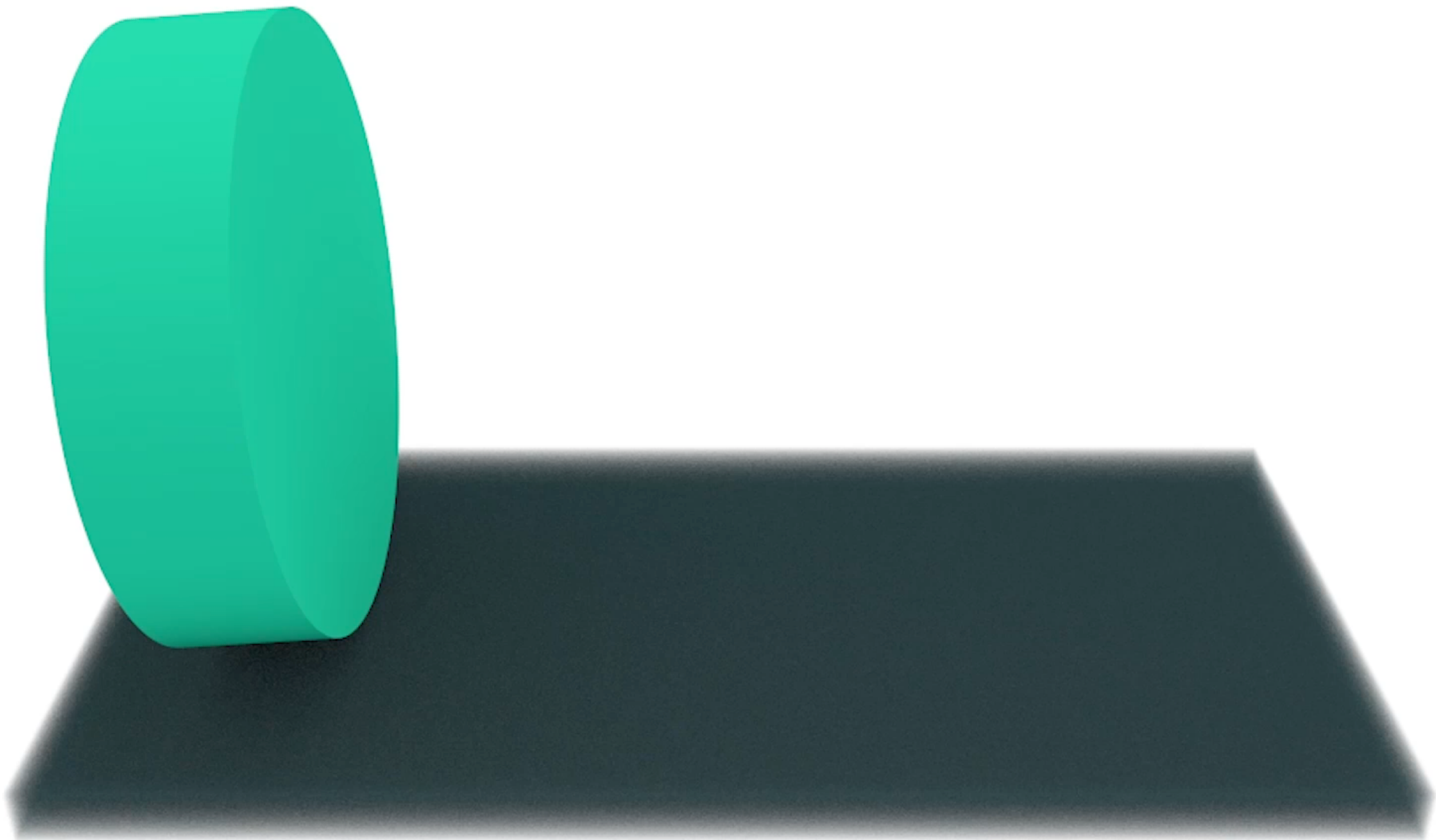


3K basis functions
[DeWitt et al.] memory: 0.76 TB
Our Memory: 1.3 GB

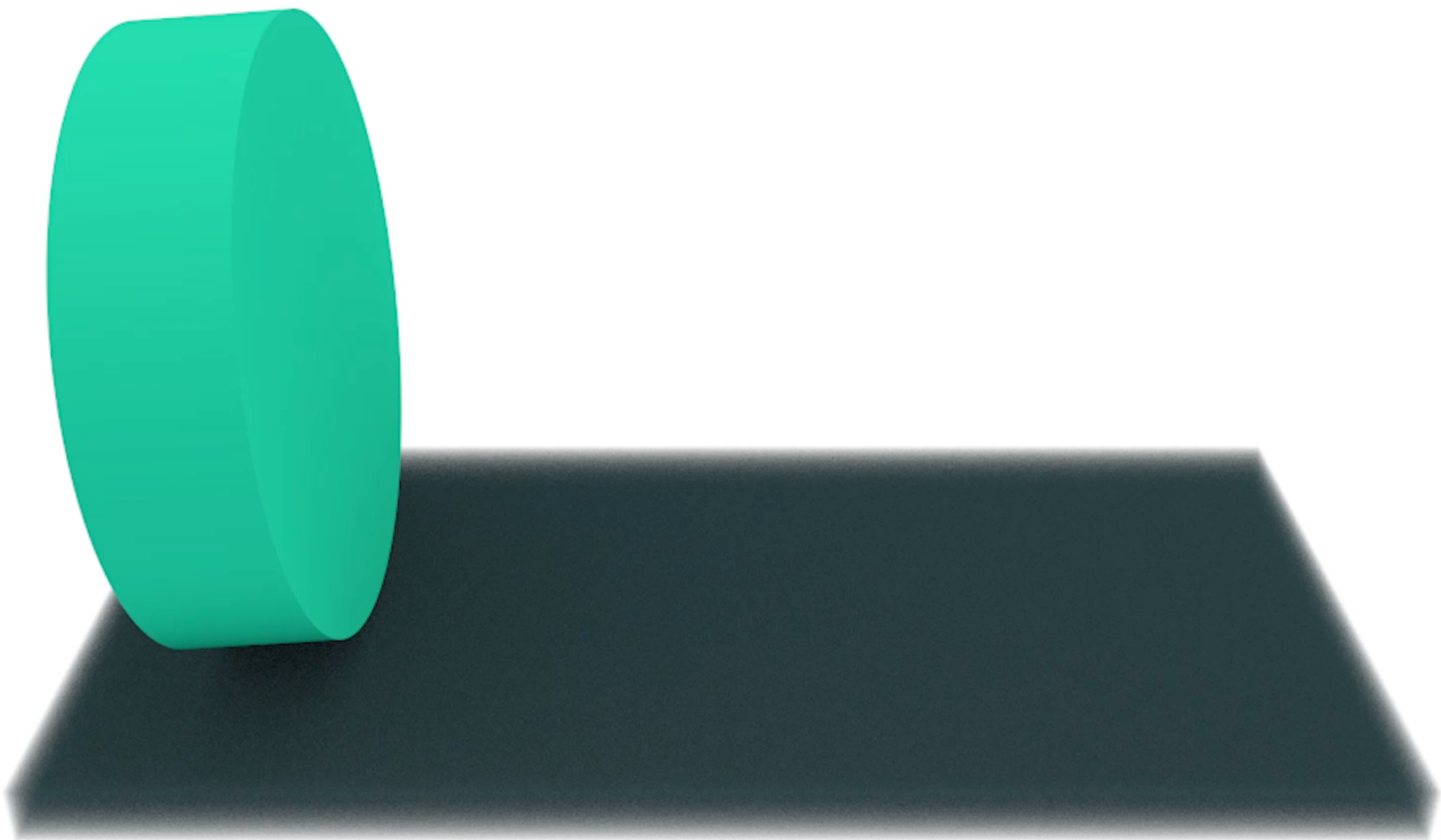


24K basis functions
[DeWitt et al.] memory: 6.1 TB
Our Memory: 26 GB

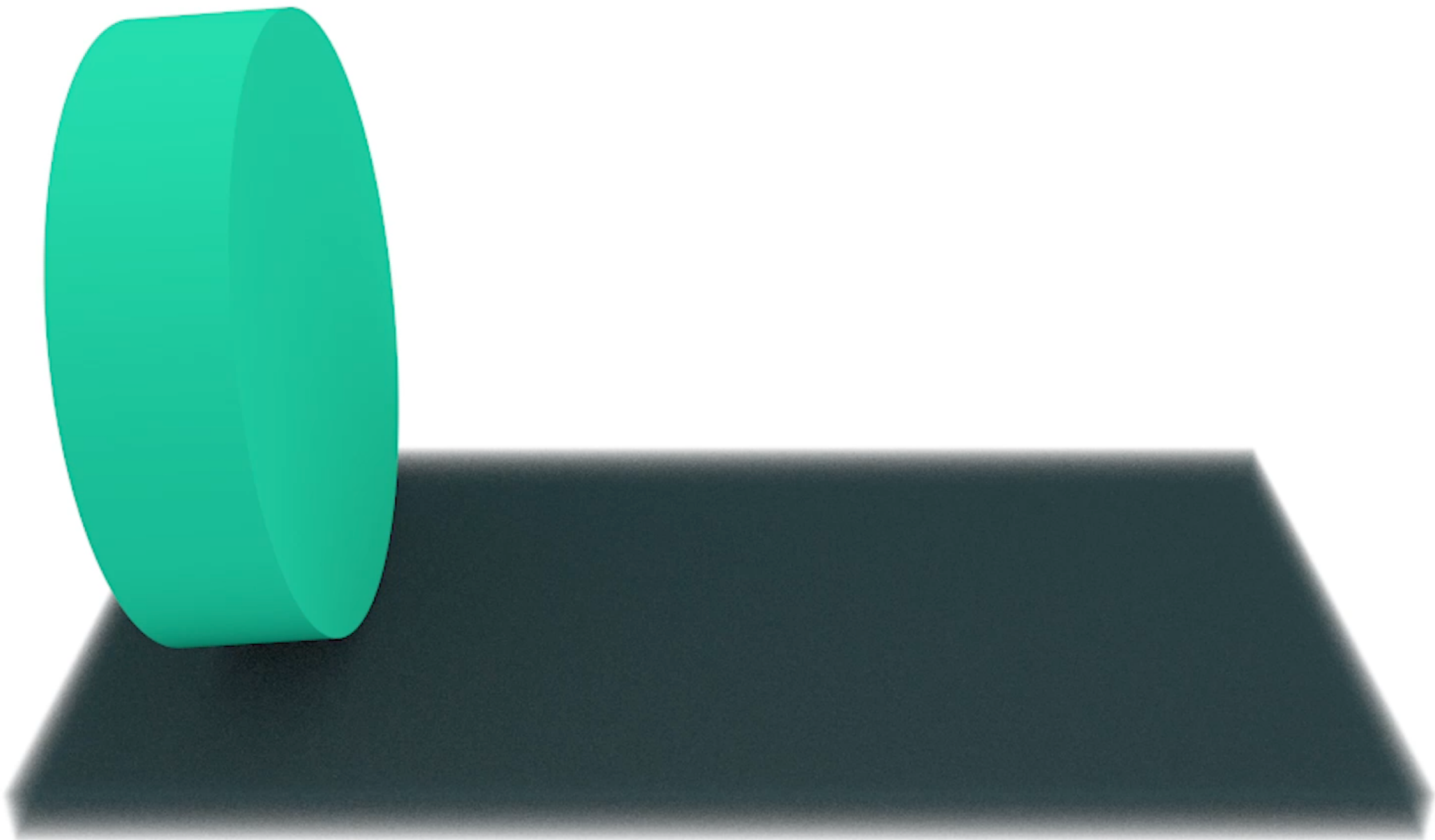
Paddle Wheel



200 basis functions
[DeWitt et al.] memory: 38 GB
Our Memory: 0.9 GB



4 K basis functions
[DeWitt et al.] memory: 0.8 TB
Our Memory: 5.1 GB



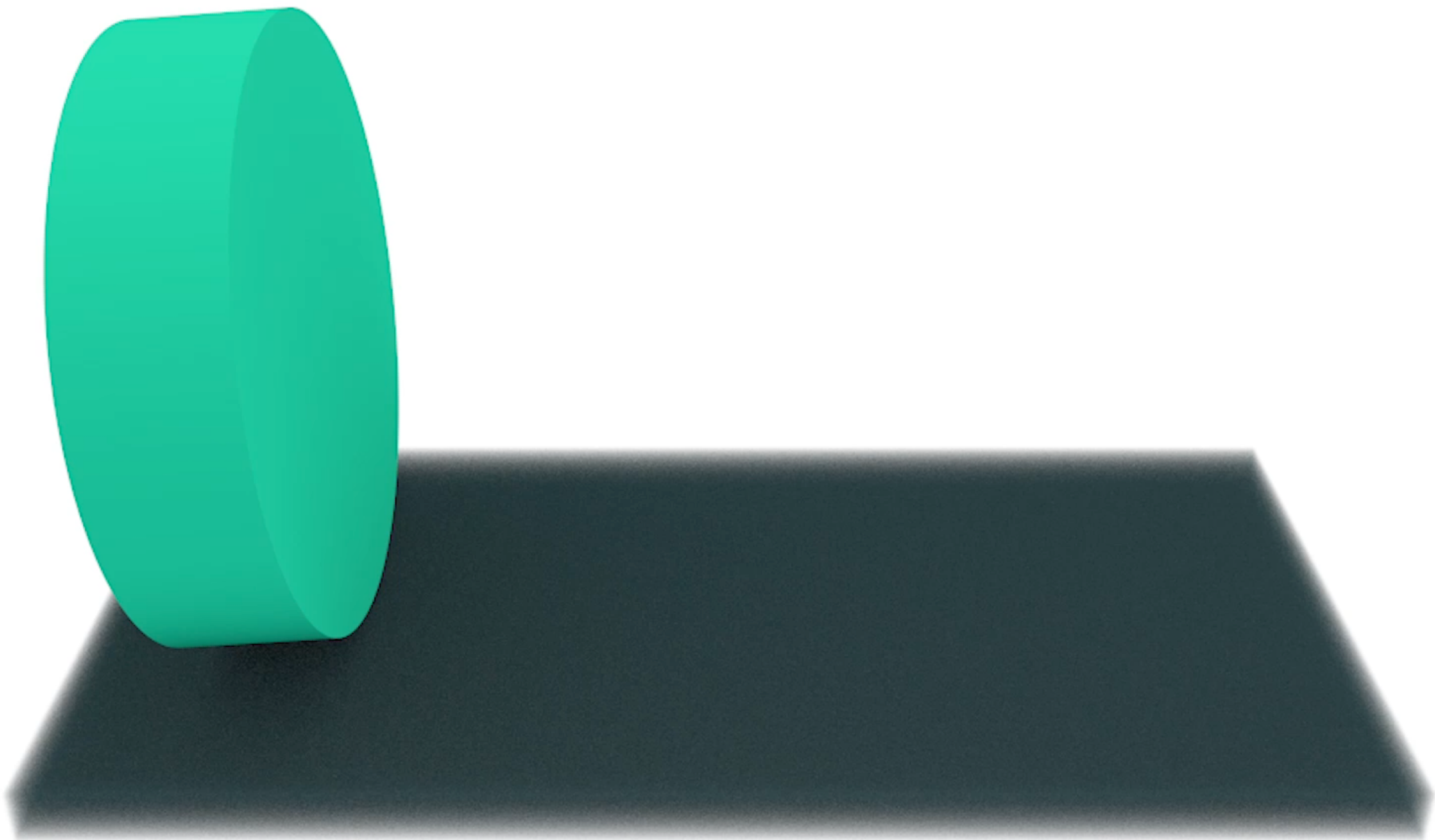
12 K basis functions
[DeWitt et al.] memory: 2.3 TB
Our Memory: 47.1 GB

Speed Comparison

128^3, 1000 basis functions

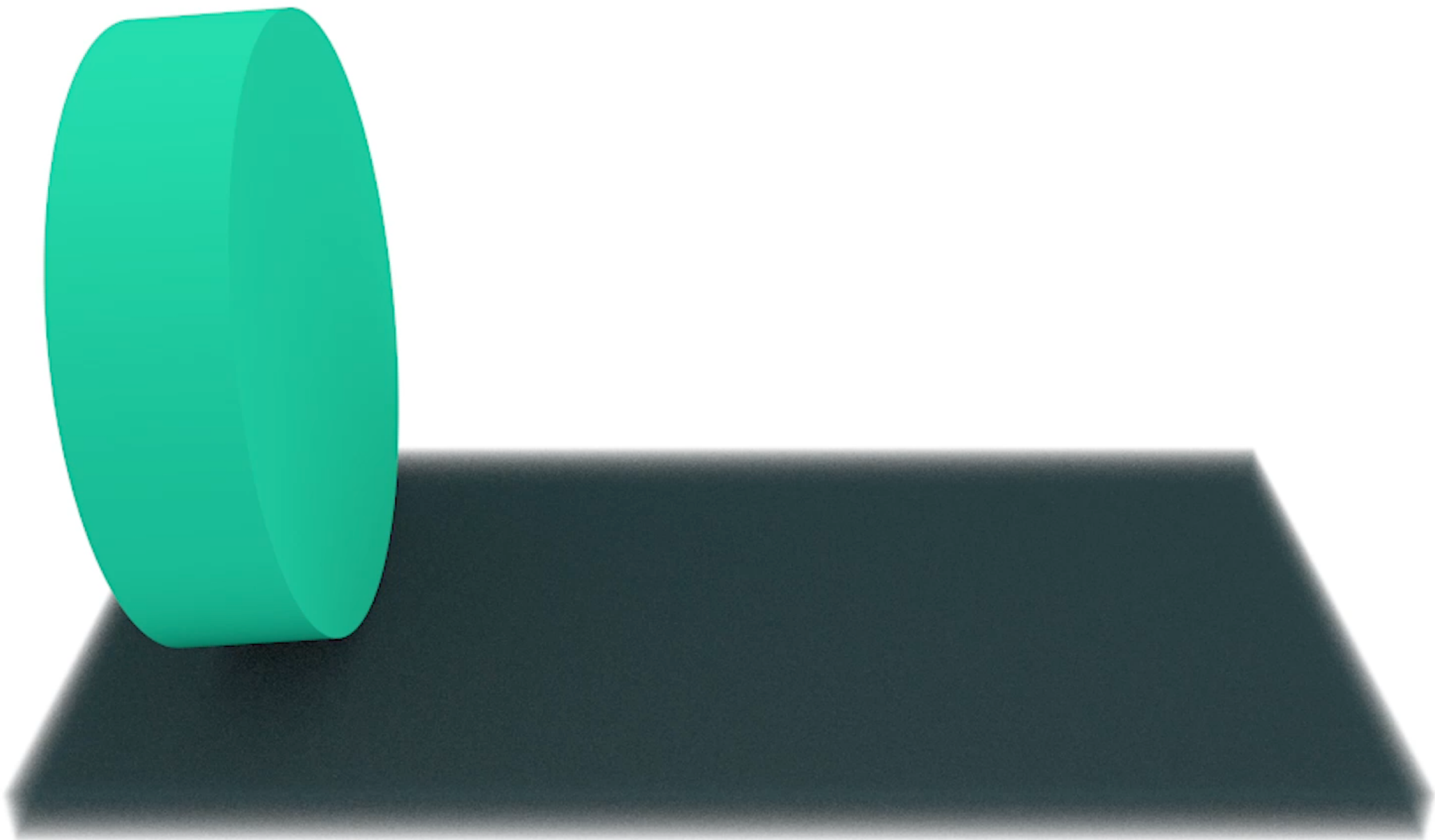
On the fly basis:	44.10 secs	440×
Cached basis:	9.54 secs	95×
Ours:	0.10 secs	

Varying Viscosity



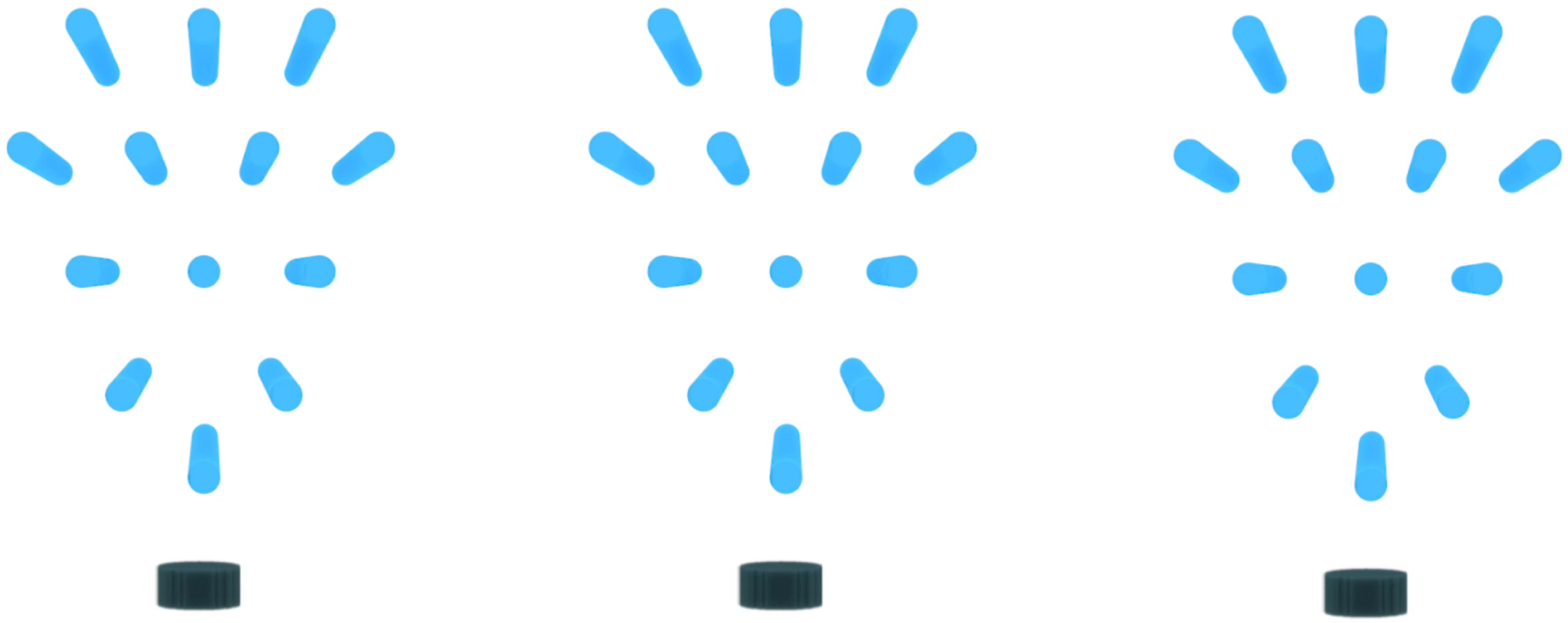
Viscosity = 0.00035

Varying Viscosity



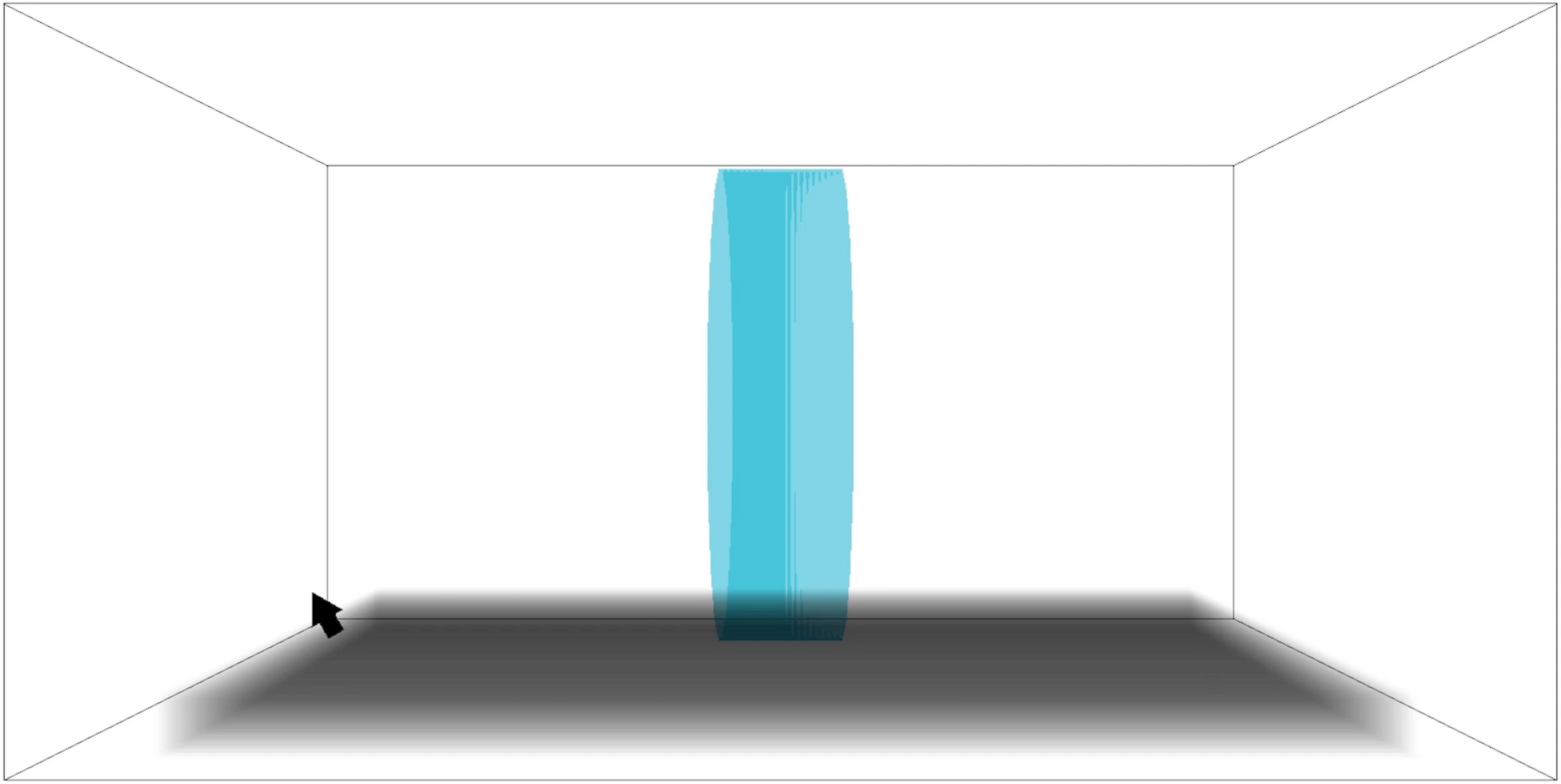
Viscosity = 0.00

Interaction with Obstacles



r = 200 r = 1000 r = 7000

Real-time Interaction



13 frames per second

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Conclusions

- Asymptotically superior in memory and time
  - Memory: $O(rN^3) \rightarrow O(r)$
  - Time: $O(rN^3) \rightarrow O(N^3 \log(N))$
- Support Neumann velocity boundaries
- Directable dynamics
- Advection tensor lossy compression
  - Up to 99.9 %

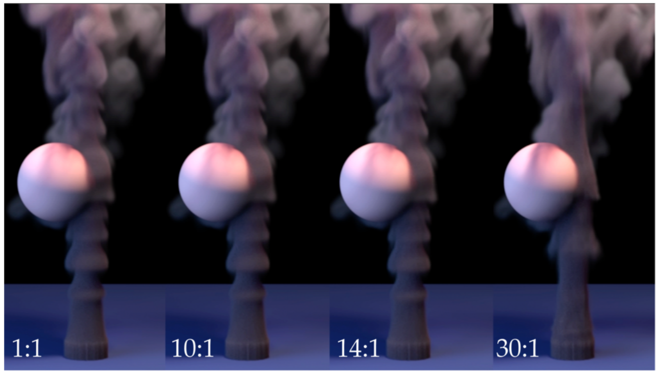
Limitations

- Rectangular domain
- Uniform boundary condition
- Energy cascade is capped at the highest frequency
- Penalty methods for obstacles

Future Work

- Advection tensor compression
- Tiled domains
- Wavelets

[Wicke et al. 2009]



[Jones et al. 2016]

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  - CAREER award IIS-1253948
  - IIS-1321168
  - IIS-1619376

- UCSB Center for Scientific Computing
  - NSF MRSEC (DMR-1720256)

Thanks!

Source code is available:

http://cvc.ucsb.edu/graphics/Papers/SIGGRAPH2018_EigenFluid/